A CONSTRUCTIVIST APPROACH TO MATHEMATICS TEACHING AND LEARNING BY A FOURTH-GRADE TEACHER

By

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To my husband, Paul,
and family--Mom, Dad, Danny, and Patricia

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Abstract of Dissertation Presented to the Graduate School
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A CONSTRUCTIVIST APPROACH TO MATHEMATICS TEACHING AND LEARNING BY A FOURTH-GRADE TEACHER

By

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Chairperson: Dorene D. Ross
Major Department: Instruction and Curriculum

The purpose of this study was to investigate what happens when an elementary teacher interprets and implements her own definition of "constructivist teaching." Constructivist learning theory has provided a basis for recent reform efforts by the National Council of Teachers of Mathematics. Constructivist learning theory has the potential to be a basis for mathematics teaching. However, there is not presently any specific model demonstrating this theory in practice. This investigation describes one model.

In order to study the teaching and learning of this fourth-grade class, the researcher used qualitative research methods to collect and analyze data. The researcher took field notes, and audiotaped or videotaped this class for 4 1/2 months. The researcher conducted several informal interviews and six formal interviews of the teacher. In addition, the teacher's plan book, resource materials, children's work, and other artifacts were examined.

Data analysis revealed several conclusions:

1. A teacher's conception of mathematics influences every aspect of the teacher's teaching--decisions about instruction, plans for instruction, assessment of students, and interaction in the classroom.

2. An elementary teacher can teach using constructivist learning principles, develop a personal teaching model of strategies, and implement these strategies.

3. Using constructivist learning strategies, a teacher can create a learning environment in which students become excited about mathematics and gain confidence in their thinking.

4. By learning to construct their own knowledge, students gain mathematical power and become mathematical thinkers.

This study presents detailed information about 10 instructional strategies that a fourth-grade teacher uses to implement a constructivist approach to teaching and learning. It also documents information about the teacher's thinking and decision making during instructional practice. While describing the classroom teaching, the researcher shows how one teacher negotiates with students to find and adapt activities for students to construct mathematical knowledge. Data show interactions between teacher and students that affect learning opportunities for students. In excerpts from interviews, the teacher discusses the dilemmas and struggles encountered when using constructivist learning principles to guide teaching and learning.

CHAPTER I
THE PROBLEM IN PERSPECTIVE

> In reality, no one can teach mathematics. Effective teachers are those who can stimulate students to learn mathematics. Educational research offers compelling evidence that students learn mathematics *well* only when they construct their own mathematical understanding. . . . There is no way to build this [understanding] except through the process of creating, constructing, and discovering mathematics.
> (National Research Council, 1989, pp. 59-60)

This study demonstrates what happens when an elementary teacher implements constructivist learning of mathematics in a fourth-grade classroom.

Introduction

Mathematics teachers, teacher educators, and researchers involved in the current reform movement in mathematics education suggest major changes in the teaching of mathematics. These people recommend that students become actively involved in constructing their own knowledge and developing mathematical concepts as they explore, explain, and justify solution strategies to mathematical tasks. The National Council of Teachers of Mathematics (NCTM) developed practical guidelines for teachers seeking to improve their teaching. These guidelines are provided in *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and *The Professional Standards for Teaching Mathematics* (NCTM, 1991). These documents deemphasize accuracy and memorization of computational procedures and emphasize teachers helping students develop mathematical power. The NCTM (1989) defined mathematical power as "the individual's ability to logically reason, conjecture, and effectively communicate about mathematics" (p. 6). Teachers help students construct mathematical power by helping them in problem solving and reasoning about mathematics and engaging them in communicating about the processes they use to reach solutions. For many teachers, this "constructivist approach" to mathematics teaching requires a change in their conceptions about mathematics and about what it means to teach and learn mathematics.

Purpose

My purpose in this dissertation is to portray how one fourth-grade teacher interprets and implements constructivist learning principles within her mathematics teaching. I will profile this teacher's practice of teaching and explain her goals. I will explain how the teacher's constructivist conceptions of mathematics teaching and learning affect her teaching. I will describe the teacher's thinking and decisions about instruction. I will describe the dilemmas this teacher encounters as opportunities are provided for students to reason, communicate, and gain mathematical power.

Theoretical Framework

Constructivist Learning Theory

The constructivist approach to teaching and learning has been emphasized in recent research in mathematics education (Cobb, 1988; Cobb, Yackel, & Wood, 1990, 1992; Kamii, 1989; Lampert, 1990, 1991, 1992). According to constructivist learning theory, students do not passively receive knowledge but actively construct new knowledge based on prior knowledge. Meaningful learning requires students' active involvement. Rather than receiving information, "the learners negotiate meaning within the context of their present understandings, make connections with past personal understandings, and modify prior knowledge in order to build new constructs" (Cobb et al., 1992, p. 6).

From a constructivist perspective, "mathematics teaching consists primarily of the mathematical interactions between a teacher and children" (Steffe & Killion, 1986, p. 207). The teacher guides the students in communicating meaning through discourse and interaction. The students then interpret and adjust this meaning to their present mathematical understandings (von Glasersfeld, 1981). Cobb and Steffe (1983) noted that "in the constructivist view, teachers should continually make a conscious attempt to see both their own and the children's actions from the children's point of view" (p. 85). The important issue then becomes one of ensuring that children construct *correct* knowledge.

Influence of Piaget and Vygotsky on Constructivist Learning Theory

Constructivist learning theory is founded in the research of Jean Piaget. Piaget (1970) referred to himself as a "constructivist" later in his life. Piaget's (1973) central concern was not whether children construct knowledge but rather the process through which they construct knowledge. For Piaget and his followers, constructivist learning is not about studying the knowledge constructed, but is about the process of constructing knowledge. Piaget suggested that knowledge is not passively received; individuals actively construct knowledge internally through their actions on objects in the world and their reflections on these actions. Building up knowledge is a process of organizing and adapting to the world. This process involves modifying old knowledge to assimilate and accommodate new knowledge.

A recent view of constructivist learning in mathematics education research adds the work of Vygotsky (1962, 1978). According to Vygotsky (1978), individuals construct knowledge in the zone of proximal development through social interactions with more knowledgeable persons. "Human learning presupposes a specific social nature and a process by which children grow into the intellectual life of those around them" (Vygotsky, 1978, p. 73). Vygotsky believed that individuals could achieve *higher ground* through talking; that is, through interaction individuals organize their thinking and actions. In this way, learning occurs two times for each individual, originally during social interaction and then again within the individual.

Recent researchers in mathematics education (Cobb, 1988; Cobb, Yackel, & Wood, 1991; Cobb et al., 1992) have asserted an interactive constructivist approach to mathematics learning and teaching. "Students come to view mathematics as an activity in which they solve mathematical problems by constructing personally meaningful, justifiable solutions while actively contributing to the interaction of the group: (Cobb et al., 1991, p. 8). My perspective of constructivist learning combines views of Piaget and Vygotsky and is similar to the perspective held by von Glasersfeld (1989) and Cobb et al. (1990, 1992). I believe that individuals internally construct knowledge; social interaction facilitates that knowledge construction. Both Piaget and Vygotsky have contributed significantly to the theoretical basis of my research. It is important to balance the cognitive and the sociological perspectives in order to understand the learning and teaching of mathematics.

Relational Understanding of Mathematics

During the past few years, I have taught inservice and preservice elementary teachers in mathematics methods courses. With my teaching, I implement constructivist learning principles in my classes. I ask my students to relearn mathematics by becoming actively involved in solving open-ended activities with several solutions. I ask them to explain and justify their thinking. I communicate to students in my classes the importance of teaching mathematics with approaches that promote *understanding*.

When I use the term “understanding mathematics,” I mean Skemp’s (1978) definition of “relational understanding.” Skemp believed that relational understanding means knowing “what to do and why” (p. 9). This type of understanding enables students to solve mathematical tasks by constructing several plans. With a relational understanding, students conceptually understand the mathematical rules and procedures and also understand the reasons for using them. Conceptual knowledge needed for relational understanding is knowledge that is rich in relationships (Hiebert & Lefevre, 1986). It can be thought of as a connected web of knowledge that connects concepts to procedures and knowledge of relationships between concepts. The relationships are parts of a network of understanding instead of pieces of information that are stored in isolation. Mathematical power is developed by students who learn a relational understanding of mathematics in which all their knowledge links and builds upon itself to form the connections for working mathematical tasks.

Research on Teacher Thinking

From my own work with preservice and inservice teachers, I have found that for them to change to teaching that implements constructivist learning theory is not easy. The process often proves difficult because of their previous conceptions of mathematics. Making changes in teaching sometimes requires changing one's conceptions of the nature of mathematical knowledge and conceptions of how mathematics is learned. My difficulties correspond with those of other mathematics educators who have suggested that essential changes in the teaching of mathematics, such as those suggested by the constructivist approach to learning, will be difficult to achieve because of what teachers believe about the nature of mathematics and mathematics teaching and learning (Brown, Cooney, & Jones, 1990; Ernest, 1989). Many elementary teachers believe that mathematics is a body of rules that should be explained to students. Because of their beliefs, many elementary teachers are unsure of how to teach mathematics in the "constructivist" ways advocated by many teachers and educators.

What are conceptions? For my research, I define conceptions as being composed of two components--beliefs and knowledge (see Figure 1). Beliefs are both affective and cognitive and are held with varying degrees of conviction. Beliefs are personal views,

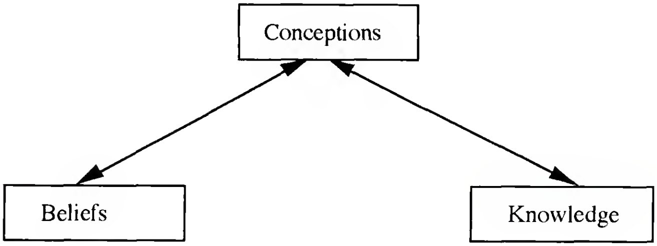


Figure 1. Components of Conceptions

assumptions, and values (Ernest, 1989). Knowledge is cognitive and must meet standards for evidence (Thompson, 1992). Mathematical knowledge is the understanding of the meaning underlying the mathematical constructs and the procedures for using them. Constructs of knowledge are the principles, rules, and mathematical statements that come from the mathematical theory of knowledge.

Teachers have particular beliefs about mathematics as a subject, about how students learn mathematics, and about how to teach mathematics (see Figure 2). Teachers may believe that mathematics should be a rule-oriented subject to be transmitted. They may believe mathematics to be full of meaningful patterns and logic that must be explained so that students can discover the theory and beauty of mathematics. They may believe mathematics is a problem-solving process in which mathematics is created by students as they discuss and interact in the classroom (Ernest, 1989). Teachers make different instructional decisions depending on their beliefs. Ernest (1991) said that the current reform movement in mathematics education is being guided by those who believe that mathematics is a changing body of knowledge created by individuals. This belief coincides with that of constructivist theorists who believe teachers of mathematics should help students to become empowered to create their own mathematical knowledge.

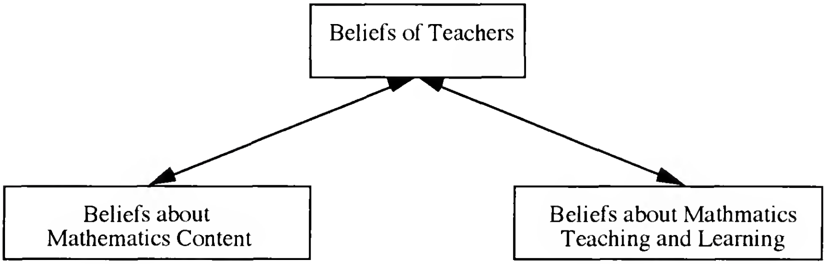

Beliefs of Teachers
Beliefs about Mathematics Content
Beliefs about Mathmatics Teaching and Learning

Figure 2. Model of Teacher Beliefs

Teacher knowledge, the second component of conceptions, includes both content knowledge and pedagogical content knowledge (see Figure 3). Content knowledge has two components, knowledge of the basic concepts and principles of mathematics rules and facts and knowledge of the ways in which mathematical truth is established (Shulman, 1986). Knowledge about teaching and learning is what Shulman identified as pedagogical content knowledge. This pedagogical content knowledge enables the teacher to choose the tasks, problems, representations, and explanations that help students understand mathematics. Part of pedagogical content knowledge is the knowledge of how children learn specific content. Teacher knowledge affects teachers' instructional decisions and classroom practice. In my research, I investigate both content knowledge and pedagogical content knowledge.

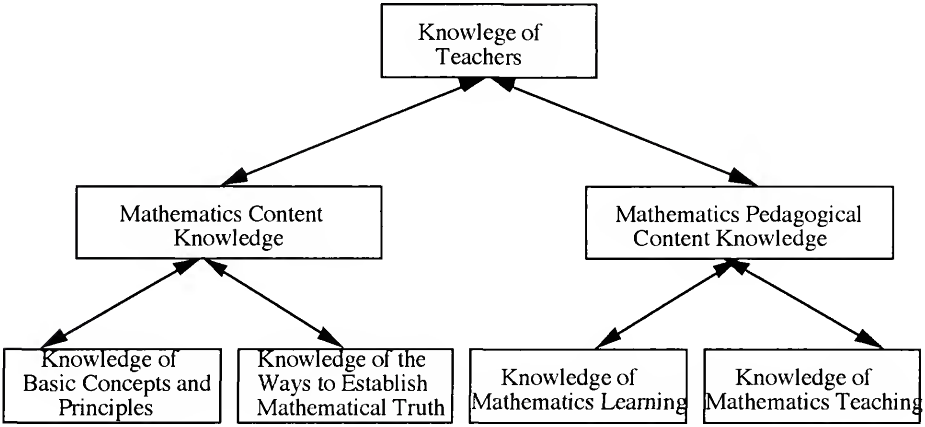

Knowlege of Teachers
Mathematics Content Knowledge
Mathematics Pedagogical Content Knowledge
Knowledge of Basic Concepts and Principles
Knowledge of the Ways to Establish Mathematical Truth
Knowledge of Mathematics Learning
Knowledge of Mathematics Teaching

Figure 3. Model of Teacher Knowledge

Thompson (1984) concluded through her research that teachers' mathematical knowledge and their beliefs about mathematics and mathematics teaching and learning affect both consciously and subconsciously their instructional decision making. Thompson

(1992) stated that teachers' beliefs about mathematics and teachers' knowledge of mathematics should be studied simultaneously. Study of mathematics teachers' beliefs of mathematics in isolation from research on mathematics teachers' knowledge obtains incomplete pictures of teaching.

Several researchers have investigated teachers' thought processes (Clark, 1988; Leinhardt, 1986, 1988, 1989; Shulman, 1986). Some observed teachers in classrooms. Others asked teachers to write reflective journals about their teaching practices. Researchers, such as Clark and Lampert (1986), asked teachers to think aloud about their classroom decisions and actions before and during instruction. Ball (1991) studied how teachers' content knowledge of mathematics affects the type of mathematics that is taught. Leinhardt (1988, 1989) investigated how teachers' knowledge of mathematics teaching affects their instructional decisions. Fennema, Carpenter, and Peterson (1989) and Carpenter and Fennema (1991, 1992) examined inservice teachers' knowledge about how children learn mathematics and investigated how this knowledge influences their instruction. All of these studies indicated that teachers do not simply behave spontaneously in the classroom but that they make decisions and solve dilemmas concerning instruction.

The progress made by these researchers in the past decade combined with past research about teacher thinking provides a strong basis for my inquiry into teacher thinking about constructivist learning principles and what happens in classrooms when teachers give children the opportunity to construct personal knowledge of mathematics.

Statement of the Problem

The NCTM (1989) stated that all students need to learn more and different mathematics than in the past. Consequently, teachers must significantly change their mathematics instruction. For students to become productive citizens in the twenty-first century, they must learn to communicate and reason mathematically (Steen, 1990). All students need access to the opportunities that the power of understanding mathematics brings. With employment for the future, we need students with the ability to think creatively and critically. We need analytical problem solvers more than people who can compute correctly (Fosnot, 1989).

Learning mathematics means more than memorizing and repeating. The NCTM (1989) stated that learning mathematics involves gaining mathematical power through investigating, representing, reasoning, and solving types of problems that we encounter in the real world. To promote the growth of students' mathematical power, teachers must select appropriate mathematical tasks and establish classroom discourse within a learning environment that encourages students to construct mathematical power. To promote the construction of mathematical power, however, teachers must be knowledgeable about appropriate mathematical tasks and tools. Equally important, teachers must believe that students construct their own relational understanding of mathematics.

Questions Guiding Inquiry

My main research question is *what happens when an elementary teacher implements constructivist learning of mathematics in an elementary classroom?* More specifically, when teachers are actively trying to implement the guidelines in *The Standards*,

1. What are the teacher's instructional decisions in
    a. lesson planning?
    b. classroom discourse?
    c. assessment of teaching and learning?
2. What does the classroom teaching look like in terms of
    a. lesson planning?
    b. classroom discourse?
    c. assessment of teaching and learning?
3. What dilemmas of teaching mathematics does the teacher encounter during instructional decision making and classroom practice?

4. How does the teacher define constructivist teaching and mathematics learning related to

   a. an examination of beliefs.
   b. an examination of knowledge.

5. What are the connections between the teacher's definitions of constructivist learning and his or her instructional decisions and classroom practice?

Using these questions, I will draw on constructivist learning theory to examine the opportunities for students to learn and understand mathematics.

Rationale and Significance

The teacher is the primary intermediary through which students learn mathematics (Ernest, 1989). Teachers assess information from and about students and the learning environment. They process this information and then decide on appropriate teaching strategies. The conceptions held by the teachers influence the processing of this information. I have designed my research in order to analyze and describe the relationship between conceptions and teaching and learning (Thompson, 1992). Any attempt to improve the quality of mathematics teaching, however, must begin by discovering how teachers' conceptions influence teacher thinking, decision making, and classroom practice.

More is known about constructivist learning than "constructivist teaching." How do constructivist learning principles guide teaching? What is a constructivist teacher? These questions have not been answered. Although a few researchers have begun to study what constructivist teaching involves (Carpenter & Fennema, 1991, 1992; Fennema et al., 1989; Cobb et al., 1990, 1992; Kamii, 1989; Lampert, 1990, 1991, 1992), few models of this type of teaching exist.

Although constructivist learning theory provides mathematics educators with useful ways to understand how individuals construct knowledge, the task of reshaping mathematics teaching based on a constructivist view of learning presents a considerable challenge. Constructivist learning theory does not tell us how to teach mathematics. However, Cobb (1988) said that a teacher with constructivist views facilitates students' construction of knowledge by negotiating with the students during interactions in the classroom. A "constructivist teacher" provides active experiences for students that encourage students to reason and communicate. This type of mathematics teaching forms the basis for my research.

Reformers in mathematics education have asked teachers to teach mathematics in ways that they never learned. Teaching in these ways is difficult. Because teachers and teacher educators are the key agents of change, they need opportunities to read about teachers whose instructional decisions and classroom teaching are helping students construct the conceptual knowledge that leads to relational understanding of mathematics. If teachers are not given the opportunities to understand what constructivist teaching involves, the vision of teaching and learning outlined in *The Standards* will become something mechanical to implement or be forgotten.

A first step toward changing mathematics teaching is for teachers to accept that problems exist within their teaching. By seeing alternative approaches to teaching mathematics, they become aware of problems that exist within their own teaching. This dissertation will increase teachers' understanding about alternative methods of teaching mathematics by describing and presenting information about decisions and instructional strategies that are used by a teacher who is constructing her own meanings of constructivist teaching. Teachers who read the study will become more aware of the strategies that they can use to help students gain mathematical power. Through this awareness, teachers may improve their ability to make effective instructional decisions that will improve their teaching practice.

Research Assumptions

All researchers begin an investigation with personal assumptions that guide their inquiry. The following assumptions underlie my research:

1. Learning is a process of conceptual change through experiencing cognitive conflict with present knowledge, assimilating and accommodating new knowledge with prior knowledge, and forming new knowledge structures (Piaget, 1970, 1980).

2. Through interactions with the world around them, people build knowledge (Schoenfeld, 1992). Learning is a social, interactive process. Through talking about understandings and communicating with more knowledgeable others, learners acquire new knowledge (Vygotsky, 1978).

3. Knowledge is "formed and transformed by the defining process that takes place in social interaction" (Blumer, 1969).

4. Students and teachers are always learning and teaching. The critical issue is what is being learned and taught (Erickson, 1986).

5. Teacher thinking is complex and influences the teacher's lesson planning (Leinhardt, 1986, 1988, 1989).

6. When teachers understand how children construct knowledge, they will use this understanding in their teaching (Carpenter & Fennema, 1991, 1992; Fennema et al., 1989).

7. Educational innovations can be understood only by analyzing the teachers' and students' interaction (Cobb, Yackel, & Wood, 1989, 1990, 1991, 1992).

Definitions of Terms

The following terms are included in and are relevant to my study:

Beliefs are personal views, assumptions, and values. Beliefs are both affective and cognitive (Ernest, 1989).

Teachers have particular beliefs about mathematics as a subject, about how students learn mathematics, and about how to teach mathematics. Teachers may believe mathematics to be a rule-oriented subject to be transmitted. They may believe mathematics to be full of meaningful patterns and logic that must be explained so that students can discover the theory and beauty of mathematics, or they may believe mathematics is a problem-solving process in which mathematics is created by students as they discuss and interact in the class (Ernest, 1989).

Beliefs about learning are teachers' personal views, values, and assumptions about how children learn mathematics. The continuum of how teachers believe students learn runs from a belief that students learn passively as teachers explain how to solve problems, to a beliefs that students actively construct knowledge while doing mathematics and talking about mathematics (Cobb, 1988).

Beliefs about teaching are teachers' personal views, values, and assumptions about how to teach mathematics. Some believe mathematics should be transmitted to students through lectures and explanations. Others believe that they are facilitators who help students construct knowledge (Ernest, 1989).

Thompson (1984) defined conceptions of mathematics as beliefs and view about the nature of mathematical knowledge. I extend this definition to define conceptions as the knowledge of and beliefs about mathematics.

Conceptual knowledge is rich in relationships. It can be thought of as a connected web of knowledge that links relationships between discrete pieces of knowledge. The relationships are as important as the pieces themselves (Hiebert & Lefevre, 1986).

Constructs of mathematics are the principles, rules, and mathematical statements that come from the mathematical theory of knowledge.

Constructivist learning theory is the process of learning where knowledge is actively created or invented by the child, not passively received from the environment. Children create new mathematical knowledge by reflecting on their physical and mental actions. Learning constructively is thought of as a process of adapting to and organizing one's own knowledge. Learning is a social process (Kamii & Lewis, 1990).

Constructivist teaching is guiding and supporting students' inventions of ideas rather than transmitting "correct" adult ways of acting. These teachers offer appropriate tasks and opportunities for dialogue. Constructivist teachers must be able to pose tasks that bring about appropriate conceptual reorganizations (Kamii & Lewis, 1990).

Content knowledge refers to the amount and organization of knowledge of the subject matter (Shulman, 1986). Knowledge of content involves the concepts and principles of the ways in which truth and validity are established in the discipline.

Pedagogical content knowledge is knowledge of teaching and learning processes. In mathematics this knowledge is needed by teachers to help students understand. It is knowledge of useful illustrations, explanations, and representations that help students understand mathematics. This knowledge is also about how children learn conceptual and procedural knowledge (Shulman, 1986).

Mathematical power is the ability to reason logically, conjecture, and effectively communicate in mathematics (NCTM, 1989).

Procedural knowledge is made up of two parts--the knowledge of formal language or symbol representations of mathematics and the knowledge of algorithms or rules for completing mathematical tasks (Hiebert & Lefevre, 1986).

Skemp (1978) said relational understanding is the understanding of mathematics that means knowing what to do and why. It is the mathematical understanding that links conceptual knowledge with procedural knowledge.

CHAPTER II
A REVIEW OF THE LITERATURE

Introduction

In order to provide a foundation from which to build my own research that demonstrates how one teacher uses constructivist learning theory in her mathematics teaching, I need to describe where the research on teaching and learning mathematics in elementary classrooms has been in the past. While I discuss past research, I will also provide a basis for why research on mathematics teaching and learning has changed.

Our democratic society's needs are constantly changing. Consequently, the mathematics that children need to know is also changing. A major goal of elementary mathematics education is to help children have the power to think about and do mathematics. This power helps students have control over their success and helps them to develop autonomy in learning. Several factors have influenced the elementary mathematics curriculum in the United States: societal forces, how children learn mathematics, and the nature of mathematics subject matter (NCTM, 1989; Riedesel, 1990). These three factors have interacted with research in psychological learning theories to influence the mathematics curriculum (Kroll, 1989; Riedesel, 1990).

Throughout most of the history of mathematics education in the United States, from the beginning through the mid-1970s, few educators or researchers made inroads into ideas of behaviorist learning theorists. Through most of those years, people believed that children learned mathematics through drill and practice. In accord with this belief, teacher effectiveness researchers primarily focused on identifying teacher behaviors, both management and instructional strategies, that would help raise student achievement. This type of research complemented the public's call for students who could perform the basic computations. The needs of society were the major factors influencing the elementary mathematics teaching.

By the mid-1980s several researchers began to investigate how children learn mathematics. Piaget's ideas of cognitive developmental learning were applied to helping children learn mathematics. Many educators came to believe that learning went beyond sequential ordering and repetition that had been derived from behaviorist learning principles. Because of this change in thinking about how children learn, within the past decade the theories informing teaching have shifted from a behaviorist theory of learning that has guided the "transmission" view of teaching toward a more cognitive view of learning--constructivist learning theory. This change in the view of how children learn has created a paradigmatic shift regarding the relationship between teaching and learning. The following review of the literature will discuss this paradigmatic shift.

Paradigms of Research on Teaching Elementary Mathematics

I include in my review of research on elementary mathematics education studies that are characterized by their depth, by their strong integration of teaching and learning of mathematics, by their impact on mathematics education curriculum, and by their representation of a continuum of research. I review four distinctive pedagogical paradigms and classify them according to where the research is focused. With the paradigms that I have developed, I demonstrate a continuum of research on teaching and learning of elementary mathematics. My review begins with a direct instruction teacher-centered model, proceeds toward a more learner-centered model of teaching with a focus on content, continues toward a constructivist learner-centered model, and then ends with a model of teaching that I call a "classroom gestalt" model. The classroom gestalt model of teaching is teacher, content, and learner-centered. I have drawn from the work of other individuals who have reviewed the literature on mathematics education (Koehler & Grouws, 1992). It is unreasonable to expect this single review of research on mathematics teaching and learning to address every issue or implication for teaching practice or future research.

Teacher-Centered Paradigm

Beginning in the late 1970s researchers in mathematics education focused on what teachers could do to be more effective in helping their students learn to compute. Researchers who studied effective teaching used the teacher-centered paradigm. They focused on the actions of teachers. I review two bodies of research within this paradigm. The first is the research conducted by Good and Grouws that focuses on the actions of teachers. The second review is Leinhardt's body of studies that focus on teacher thinking.

Good and Grouws used the research paradigm called "process-product." These studies focused on teacher behavior (process) that affects student behavior which in turn affects student performance or achievement (product). Mathematics education researchers built upon the experimental and correlational research of others (Brophy, 1979; Gage, 1978; Rosenshine, 1976) in answering questions about what effects teachers' instruction had on student achievement. These researchers identified behaviors of effective teachers associated with student achievement.

By observing classroom teachers, Douglas Grouws, a mathematics educator from Missouri, became interested in the effects of teacher behavior on student mathematics learning (Good & Grouws, 1979). Good and Grouws studied over 100 third- and fourth-grade teachers who taught in the same school district. They used student outcomes on standardized achievement tests in mathematics to compare teachers' effectiveness. They observed these teachers for over 3 years and compared test scores. The researchers found that some teachers produced more mathematics learning than other teachers who were using the same textbook and were teaching students with comparable abilities.

To follow up this study, Good and Grouws began to observe teachers to find the behaviors and instructional strategies that were associated with high and low achievement of students. They found that teachers differed greatly in their classroom behavior. More effective and less effective teachers taught very differently. Teachers who obtained higher levels on student achievement provided students with a clearer focus of what they wanted their students to learn. These teachers provided more time for students to practice new procedures with immediate teacher feedback. This time was called the developmental portion of the lesson. Good and Grouws claimed that the developmental time that teachers gave to students affected achievement. From these studies, Good and Grouws hypothesized that certain teacher behaviors affected higher or lower achievement of students. The teacher behaviors were congruent with the direct instruction model advocated by other researchers in education (Rosenshine, 1978). The model included daily review, development of the lesson by lecture and teacher demonstrations of procedures, supervised practice, independent practice, and conclusion of the lesson. Using this model, Good, Grouws, and Ebmeier (1983) taught teachers the instructional behaviors associated with high student achievement.

In an experimental study called the Missouri Mathematics Program, Good et al. studied the effectiveness of their teacher training program. They assigned half of 40 classrooms to the experimental group, and the other half to the control group. The experimental teachers were taught the teacher behaviors that the researchers believed increased student achievement.

Good et al. gave standardized achievement tests as pre- and posttests. After 2 1/2 months, students in the experimental classrooms performed substantially better than those in the control classrooms. By collecting data from the regular end-of-the-year achievement tests, the researchers found that students taught in the experimental classrooms continued to achieve higher test scores than students in the control classrooms.

Good et al. observed both experimental and control classrooms to record the frequency of particular teacher behaviors that influenced student achievement. They studied how teachers sequenced lesson segments: the introductory segment, the developmental segment, and the conclusion. From these observations, Good et al. developed a model of teaching called *active teaching*. Active teachers increased the amount of time that students practiced and experienced success on skills before being tested. They evaluated whether the students were performing accurately. These teachers retaught the content if necessary. The teachers in the experimental classrooms used whole group instruction to introduce concepts, demonstrate procedures, and provided drill and practice for students with immediate teacher feedback.

One of the findings from the Good et al. study was that the teacher should present the lesson in small steps with students practicing after each step. The teachers guided the students through frequent successful practice. The researchers concluded that teacher actions did make a difference in student achievement. Their findings further demonstrated that teachers could be trained to provide instruction that would increase student achievement.

The findings from the Missouri Mathematics Program that teachers do make a difference in student learning spurred educators to investigate how these *active teachers* planned and organized their lessons. They identified a model of teaching that could be used to improve performance regardless of content. Even though this type of research was teacher-centered, it had not addressed all of the issues concerning the complexities of teaching. The researchers did not ask questions such as the following: How does the teacher plan what to teach? What is the teacher's thinking during instruction?

Teacher thinking about teaching--before and during lessons--became an important question and began to play a role in research on effective teaching (Clark & Peterson, 1986). Clark and Peterson suggested in their research on teacher thinking that teacher planning and interactive teaching are influenced by the teacher's reflection. They concluded from classroom observations and interviews with teachers, both before and after their lessons, that teacher thinking is influenced greatly by the teacher's beliefs and knowledge about teaching. Therefore, teacher thinking had a powerful effect on what and how children learn.

Gaea Leinhardt (1986, 1988, 1989) compared expert and novice teachers' performance to study how teacher thinking affects classroom teaching. She examined the teachers' planning and decision making before lessons and during teaching. Leinhardt's goal was to develop teacher education programs that made novice teaching more like that of the experts.

Leinhardt and Smith (1985) explained that the cognitive skill of teaching draws on two types of knowledge: lesson structure knowledge and subject matter knowledge. Lesson structure knowledge includes "the skills needed to plan and run a lesson smoothly, to pass easily from one segment to another, and to explain material clearly" (p. 247). Subject matter knowledge includes knowledge of "concepts, algorithmic operations, the connections among different algorithm(s) . . . understanding of . . . student errors, and curriculum knowledge" (p. 247).

Leinhardt and Greeno (1986) observed eight expert and four novice fourth-grade teachers while teaching lessons on fractions for 3 1/2 months a year for 3 years. They videotaped the teachers' lessons and interviewed the teachers about planning and evaluation of their lessons and fraction knowledge. Experts had more subject matter knowledge of fractions. They connected the concept of fraction--a part of a whole--with the procedures. Expert teachers understood how the operations on fractions--addition, subtraction, multiplication, and division--were connected to each other. Because novice teachers' knowledge of fractions consisted of isolated concepts rather than connections among concepts, they separated addition and subtraction of fractions into two different types of problems. Novices did not see the connection that the concept of fraction brought to the two types of problems. Novice teachers lacked conceptual knowledge of fractions.

In a similar project Leinhardt (1986) studied elementary teachers to find out what made an expert teacher. She studied seven experienced teachers. Again, the researchers observed and videotaped teachers during mathematics lessons. They interviewed teachers about their views of pedagogy and math knowledge. Leinhardt found that expert teachers' lessons had clarity and a coherent focus on content. Because experts possessed more lesson structure knowledge, they introduced and represented procedures differently from novice teachers. The expert teacher presented clear explanations of each procedure and demonstrated several methods of approaching a problem. Leinhardt (1988) concluded that "experts are unusually good at constructing series of lessons that successfully transmit the content that needs to be learned" (p. 29).

Leinhardt (1989) explained that teaching a lesson is a complicated process that consists of three basic components: a lesson segment, an explanation, and an agenda. A lesson segment is an overarching framework that guides each lesson. Each segment has a unique goal, actions, and task environments. The second lesson component, the explanation, is the "actual transmission of the subject matter content" (p. 56). Leinhardt stated that the explanation of the lesson should be the most carefully constructed segment because it presents new mathematical material. The explanation segment is equivalent to the developmental portion of the lesson discussed by Good et al. (1983). The last component of a lesson is the agenda. Leinhardt (1989) defined an agenda as "a unique operational plan that a teacher uses to teach a mathematics lesson. It includes both the objective or goals for lesson segments and the action that can be used to achieve them" (p. 55). Leinhardt noted that teachers' agendas are not formal written lesson plans but mental notes about what they expect to happen during the lesson.

In her research, Leinhardt (1989) found contrasts in the lesson planning of expert and novice teachers. Novice teachers' agendas followed a structured script that was often incomplete and did not include actions of the students. Experts' lesson segments had patterns. Expert teachers planned agendas that were rich in content. Leinhardt concluded that expert teachers weave lessons together. Their lessons begin with a good presentation of content. This presentation affects student learning and performance.

Findings from Leinhardt's research (1986, 1988, 1989) are that expert teachers' lessons are explicit and direct much like the active teaching model of Good et al. (1983). Expert mathematics teachers design lessons that are well sequenced and connected to previous lessons. Their lessons have closure that binds the concepts and patterns together. Experts spend less time in transition between lessons.

Leinhardt (1986, 1988, 1989) extended process/product research in mathematics beyond its focus on specific instructional behaviors that affected students' outcomes on achievement tests to a focus on teacher thinking. She furthered the research of Good et al. (1983) and laid a foundation for further inquiry into cognition. This foundation opened the door for others to investigate student thinking. For example, what are the effects of student thinking on student achievement? In the next section, I review the literature that demonstrates a paradigmatic shift from research on teacher behaviors and thinking to a focus on how students think about content.

Learner-Centered Paradigm with a Focus on Content

Researchers using this approach for studying teaching examined student thought processes. They believed that student thinking is the mediator between teaching and student learning. I review two bodies of research within this paradigm, investigations of Hiebert and colleagues and Cognitively Guided Instruction developed by Carpenter and Fennema at the University of Wisconsin. Both bodies of research focus on how children learn specific mathematics content.

In 1985, Hiebert and Wearne began gathering data on students' knowledge of decimals. They chose 700 students in grades 3, 4, 5, 6, 7, and 9. In the fall and spring of the school year, they gave students written tests on computing, changing, and ordering decimals. They interviewed 150 students two or three times during the year. The purpose of the interviews was to evaluate students' conceptual knowledge about decimals. They wanted to find out if students used their conceptual knowledge to help them solve problems during written tests. One of their most significant findings was that students did not use their conceptual knowledge of decimals to perform the procedures.

Hiebert and Wearne (1986) concluded that the students' lack of connection between the concepts and the calculations was so clear that they could be said to possess two separate kinds of knowledge. They defined one type, conceptual knowledge, as "knowledge that is rich in relationships, . . . thought of as a connected web of knowledge . . . [and] cannot be an isolated piece of information (Hiebert & Lefevre, 1986, p. 4). They defined the second type of knowledge, procedural knowledge, as knowledge of symbols and rules for working out algorithms, the "step-by-step instructions that prescribe how to complete tasks" (Hiebert & Lefevre, 1986, p. 6). Most students could only solve routine problems that involved procedural knowledge. When they tried to use conceptual knowledge to solve nonroutine problems, they invented procedures that were profoundly incorrect. Students' performances on the tests were so governed by rules of procedures that Hiebert and Wearne (1985, 1991) were able to develop models that illustrated the students' behavior in solving decimal problems.

The researchers decided that without conceptual knowledge, students memorized large numbers of rules for symbols and algorithms. When the conceptual knowledge was lacking, students distorted the rules because they were unable to keep all the memorized rules categorized. Students who performed procedures often did not understand the procedures (conceptual knowledge) but had simply been good at memorizing all the symbols and rules for the procedures (procedural knowledge). Because students performed far beyond their ability to understand, Hiebert and Wearne (1986, 1991) concluded that learning only procedures can sometimes be too powerful for students.

Based on their earlier research, Hiebert (1988, 1992) theorized that there were three key cognitive processes that individuals needed to learn a specific content. Key cognitive processes are the processes that enable students to understand new knowledge. The clearest way to identify key cognitive processes is to describe their characteristics. One characteristic of key cognitive processes is that students need to construct content by connecting new information to other knowledge that they already possess. A second characteristic of key cognitive processes is that students must be able to reflect upon the processes and make sense of using them. This reflection makes the process of learning content something like higher-order thinking. A third characteristic is that students must be able to transfer new key cognitive processes to new problem-solving situations. For example, adding decimals to the tenths place can be generalized to adding them to the hundredths and thousandths place. Hiebert and Lefevre (1986) wrote that procedures must be linked with conceptual knowledge so that students can store the knowledge as part of a network of information. Hiebert and Wearne (1991) stated that the most important component of research on teaching is to identify and study the key cognitive processes involved in learning.

This research (Hiebert & Lefevre, 1986; Hiebert & Wearne, 1985, 1986, 1991) on learning built a foundation for their next step--applying knowledge of learning to instruction. The researchers chose a particular content domain, decimal fractions, and identified the cognitive process involved in learning the content. They designed instruction and then examined the extent of the change in students' knowledge of the content. Hiebert and his colleagues designed instruction in decimals that would help students connect procedural and conceptual knowledge. In nine lessons, they taught 9 fourth-graders and 10 fifth- and sixth-graders to add and subtract decimals. The fourth-grade students had not had previous instruction in adding and subtracting decimals. In the first five lessons, students used Dienes base-10 blocks to learn the conceptual knowledge of adding and subtracting decimals. In the last four lessons students learned the symbols and manipulated numbers to learn the procedures for adding and subtracting decimals. Hiebert and Wearne (1985) interviewed the students before the instruction and 6 weeks after the instruction.

Hiebert and Wearne's (1991) major findings from this study were that students can learn to use models to represent written symbols and operations, students can develop symbolic procedures from representations with models, and students can transfer new knowledge to new problem situations. Perhaps the most significant finding was that prior instruction that emphasizes manipulating numbers and memorizing rules interferes with learning conceptual knowledge.

After this study, Hiebert and Wearne (1991) designed a study to determine whether results would be similar with large-group instruction. The subjects for this study were students in two classes, a fourth-grade and fifth-grade class. With only slight modifications in the sequence of the lessons, the researchers taught students to add and subtract decimal fractions. On five separate occasions, researchers assessed student achievement through group tests--before, during, and after instruction. Hiebert and Wearne also chose six students to interview before and after instruction. The findings were similar to the previous small-group instruction study except that fewer students transferred their knowledge to new situations. In addition, fourth-graders benefited more from the researchers' instruction than the fifth-graders. This finding confirmed data from their previous study that prior rote learning interfered with the acquisition of conceptual knowledge. Hiebert and Wearne concluded that it is better for students to develop meanings for symbols before they practice computations.

Hiebert (1992) stated that instruction must be specifically designed to help students build connections between procedures and concepts. He believed that the most important features of instruction are (a) explicit attention to connections between meaningful familiar manipulatives and the symbols and operations, (b) time for student reflection on the meanings, and (c) the development of a rich language that can be used to talk about symbols.

Hiebert and his colleagues' research helped mathematics educators to become aware of the need to connect research on teaching with research on learning. Even though their research suggested that the students' performance is based upon their own actions and behaviors, these researchers continued to believe, like Good et al. (1983), that the teacher influences the students' actions. This research indicated that teachers should carefully structure instruction to lead students step-by-step through learning the key cognitive processes. Good, Grouws, and Ebmeier, Leinhardt and colleagues, and Hiebert and

Wearne all emphasized the need for teachers to organize lessons so that new mathematical knowledge is presented in small steps for the students to learn.

The following approach to research is similar to Hiebert and Wearne's (1985, 1986, 1991) research on how children learn decimal fraction content in that both models show that teaching can become more effective by studying how students learn specific mathematics content. Fennema, Carpenter, and Peterson (1989) began their research with the goal of providing teachers with the knowledge of how children learn mathematics. After giving teachers information on how children learn mathematics, they studied how teachers used this new knowledge. Fennema et al. (1989) hoped to change the teachers' traditional beliefs about how children should learn mathematics. However, instead of providing a script for lessons for teachers as did Hiebert and Wearne (1991), Fennema et al. (1989) wanted teachers to use knowledge about how children learn mathematics content in their classrooms and to reflect upon its consequences for their students in learning mathematics.

In the early 1980s, Carpenter and Moser (1984) began studying how children learn to add and subtract in informal situations. These researchers found that children invent strategies for solving addition and subtraction problems. Children may add on, count back, compare groups, or derive facts. The researchers found that these strategies were quite different from the way children are taught formal mathematics in the early grades. In these grades, children often are taught how to add with symbols (e.g., 6 + 7) before they conceptually understand the relationship of the two numbers or the concepts of addition and subtraction. By using word problems, the researchers found that children naturally solve addition and subtraction problems. Researchers concluded that children come to school with informal knowledge about how to add, subtract, multiply, divide, and work with fractions.

Using the previous research, Carpenter and Fennema (1991) and Carpenter, Fennema, and Franke, (1992a, 1992b, 1992c) developed a hierarchy based on children's informal knowledge that explains children's solution strategies for particular problem types. Throughout their research, they asked students to explain verbally their solutions to problems. Fennema et al. (1989) and Carpenter and Fennema (1991) concluded that for symbols and computational algorithms to have meaning for children, they must be clearly linked to what the children intuitively understand. If children do not make the connections between the concepts and the procedures, they will have only a rote understanding of the procedures and algorithms they may be taught. Without the conceptual knowledge, if a child forgets one of the rules, he or she will have no basis to figure out what to do with a problem.

Fennema et al. (1989) and Fennema, Carpenter, and Peterson (1991) stated that the key to constructing knowledge in mathematics is for children to describe their modeling or counting solution strategies verbally. When children are allowed to think and talk about problems, they begin to think about problems in more abstract terms. When children are able to think more abstractly, teachers can help provide the transition to symbolic procedures. Based upon their research on how children think, Fennema and Carpenter with other educators trained teachers to facilitate cognitive learning in mathematics. This program was named Cognitively Guided Instruction.

The Cognitively Guided Instruction (CGI) project began as inservice workshops for first-grade teachers to help them understand how children develop addition and subtraction concepts (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989). Carpenter et al. developed a model that integrated cognitive science and curriculum development. Their model made teachers' and students' thinking the central focus. In the first year of implementation, 40 teachers were randomly assigned to either an experimental or control group. The experimental group of 20 first-grade teachers participated in a month-long summer workshop. In this workshop the teachers learned to classify addition and subtraction problem types and children's solution strategies for different problems. Teachers discussed how this knowledge might affect their assessment of their students thinking and their planning for instruction. The researchers did not prescribe instructional practices during the workshop.

Based upon their research, Carpenter et al. found that in classes where teachers used CGI, students not only learned the concepts but also were better with computations. The CGI teachers encouraged students to use a variety of problem-solving strategies, and they listened to processes their students used significantly more than teachers that had not participated in the CGI workshops. In addition, CGI teachers knew more about individual students' thinking during problem solving.

The CGI teachers, more than "non-CGI" teachers, came to believe that instruction should build upon students' informal prior knowledge. Even though teachers in CGI classes spent half as much time explicitly teaching number facts as the non-CGI teachers, students in CGI classrooms recalled number facts at a higher level. The CGI students also surpassed the students in non-CGI classes in problem-solving abilities, understanding concepts, and confidence in their mathematical abilities.

Researchers such as Hiebert and Wearne and Carpenter and Fennema address issues concerning how children learn mathematics and help continue the reform movement in elementary mathematics education that began with the *Agenda for Action* (NCTM, 1980) which focused on problem solving. These researchers increase our understanding of how children learn mathematics. Through this knowledge, researchers give teachers opportunities for building bridges between the informal mathematical experiences of the child and formal mathematical knowledge of the expert, the knowledgeable teacher.

Through the research of Hiebert and Wearne (1985, 1986, 1991) and Carpenter and Fennema, we have learned that children often do not construct a relational understanding of mathematics through the behaviorist approach where teachers transmit facts, skills, and concepts to the students. Consequently, researchers in elementary mathematics began to study cognitive approaches to learning.

Constructivist Learner-Centered Paradigm

The constructivist learner-centered paradigm represents a definite shift from teacher-directed instruction to learner-oriented thinking. The role of the teacher changes from one of transmitter to one of facilitator. Children help make the decisions about topics to be learned and about how learning will take place. This body of research is similar to Cognitively Guided Instruction because its focus is on children constructing knowledge. However, the learner plays a more active part in deciding what happens in the classroom.

Constructivist learning is a process by which an individual actively creates and invents new knowledge. Educators who advocate constructivist learning in mathematics say that children create new mathematical knowledge by reflecting on their thinking and actions while they solve problems. Most constructivists trace their origins about learning back to Piaget (1970, 1980). They believe that people learn through a process of individually acting on the world. Piaget said that during the process of *disequilibrium*, individuals internally experience cognitive conflict when confronted with new information. During disequilibrium, prior knowledge cannot explain new experiences. Therefore, through accommodating new knowledge and assimilating it with the prior knowledge, individuals form internal structures of knowledge unique to them (Fosnot, 1989; Piaget, 1970, 1980; Schoenfeld, 1992). Piaget (1970) claimed that interactions in the classroom can facilitate knowledge development because interaction can create cognitive conflict that can change thinking. Piaget claimed that peer interactions stimulate student reflection about ideas that other students present.

Also contributing to constructivist learning theory in mathematics are the ideas of Vygotsky (1962, 1978). Vygotsky suggested that individuals construct knowledge in the zone of proximal development through social interaction with more knowledgeable others. During interactions with others, individuals learn as they communicate their thinking. Learning happens when individuals construct their own interpretations through language. Vygotsky (1978) noted that social interaction not only initiates changes in thinking but also alters current thinking. Individuals gradually internalize the talk that occurs during interactions. "Any function in the child's cultural development appears twice or on two planes. First it appears on the social plane, and then on the psychological plane. . . . Social relations . . . underlie all higher [cognitive] functions and their relationships" (Vygotsky, 1978, p. 57). With this statement, Vygotsky explained the internalization of knowledge as both a social and a psychological activity.

Cobb, Yackel, and Wood (1990) and von Glasersfeld (1989) have developed a way of thinking about constructivist learning that combines the ideas of Piaget and Vygotsky. Learning is a social and cognitive process in which children share their thinking. Both perspectives, the cognitive and the sociocultural, are essential in the learning process and they complement one another (Cobb, 1994). This line of thinking is called *social constructivism* (Cobb et al., 1990). It puts great emphasis on communicating and negotiating as a process for constructing knowledge. In the mathematics classroom, teacher and students continually use each others' contributions to resolve disequilibrium and develop individual knowledge. However, during the process of negotiating and sharing with a knowledgeable teacher, students come to understand the mathematical meanings of the wider society, *taken-as-shared-meanings* (Cobb et al., 1991). Students adapt to the actions of others in the course of ongoing interactions. During group discussions, individuals construct their own knowledge as they make sense of others' ideas and resolve disequilibrium caused by differences between their ideas and those of others.

Teachers who strive to help children construct knowledge of mathematics often change their roles as teachers. A teacher becomes a mediator between students' personal understandings of knowledge and the culturally established mathematical meanings (Cobb, 1994). Teachers guide and support students' thinking rather than act as the authority for correct answers (NCTM, 1991). The students must decide on the correctness of their knowledge constructions. When children are constructing their knowledge, teachers must be able to pose tasks that help children construct meaningful conceptual knowledge upon their prior understandings.

To illustrate the learner-centered paradigm, I discuss the research of Paul Cobb and his colleagues and Constance Kamii as illustrative of the paradigm. Cobb, Yackel, and Wood (1989, 1990, 1991, 1992) built upon the research of Piaget. However, they added a new dimension: students learning in cooperative groups. Also, their work is greatly influenced by Vygotsky (1962) who believed that learners must internalize knowledge while constructing it through social interaction. Cobb et al. believed that students need to learn in a setting where they can share their ideas. Small groups provide this setting.

Cobb and colleagues began their research in a second-grade classroom in 1988 with a classroom “teaching experiment.” This experiment was similar to the constructivist teaching experiment developed by Cobb and Steffe (1983) in which a researcher worked with and interacted with one child or a group of children to investigate understandings and development of knowledge. In this teaching experiment, researchers guided the learning to study whether certain materials or activities were useful for students to construct conceptual knowledge.

In their second-grade teaching experiment, Cobb et al. (1990, 1991) videotaped and interviewed the students for 1 year in order to develop instructional activities that would cause students to experience conflict with their present knowledge and resolve problems by constructing their own solution strategies. The researchers used earlier models of number learning developed by Steffe, von Glasersfeld, Richards, and Cobb (1983) to help them develop activities. They modified the activities day-by-day as they interviewed the students and analyzed the students’ solution strategies for mathematical tasks.

During the spring of the first year, the researchers met once a week with the teacher. They asked her to reflect on her teaching. At first, the teacher did not find her teaching problematic and was reluctant to think about changing her “transmission” teaching approach. She thought that her children were learning what she was teaching them. The researchers showed her videotaped interviews of some of her students solving problems. These videotapes showed that the students did not have conceptual understanding of the content that she had taught them. Nevertheless, she remained unconvinced that her teaching was a problem. The researchers then asked her to interview and evaluate two students in her class. During her evaluation of their place value understanding, the teacher became convinced that her students did not understand place value and had merely memorized the algorithms. As a result of these interviews, the teacher became willing to work with the researchers to help her to develop an environment where her students would have opportunities to learn mathematics through activities that addressed both cognitive learning and social interaction.

Also, in the second year of their study, the researchers videotaped and observed the same teacher with 20 different students. Cobb et al. (1990, 1991) daily videotaped and interviewed eight students who were paired in cooperative groups during mathematics class. Each day they reviewed the videos in order to develop and modify instructional activities. The teacher used two types of instruction: whole-class discussion and small-group problem solving. The teacher began class by introducing each lesson for 5 minutes. She made sure that the students understood directions and any symbols needed for the cooperative-group activities. The teacher did not explain or demonstrate any procedures or solutions. After the 5-minute introduction, the children worked in pairs for approximately 25 minutes. They cooperatively solved problems and activities developed by the researchers.

During each mathematics class, the teacher gave the pair of students one activity page to complete. If the pair finished the activity, they could get another one. Some groups finished four or five activities a day. The teacher emphasized, however, that thinking meaningfully about the mathematics was more important than finishing more than one activity. The teacher always circulated among the groups observing the students and intervening if they had trouble cooperating.

After students worked on the cooperative activities, the teacher then reassembled for whole-class discussions. At first, the teacher had difficulty not telling the students when an answer was right or wrong. However, by the end of the year, she was comfortable in her new role. Whether students' answers were correct or incorrect, the teacher asked the students to explain and justify their solutions. Sometimes she would help them to reconstruct or explain their answers. The teacher became responsible for helping other children understand what each child meant during an explanation or an attempt to solve a problem. Rather than evaluating the correctness of answers herself, the teacher asked the students to decide if an answer was correct. The teacher concentrated the discussions on small parts of the activities so that discussions were in-depth. The teacher had to work to develop an environment where students felt comfortable in taking risks to explain how they solved problems.

As the year progressed, the teacher evolved and constructed her knowledge of her students and how they learned. As she reflected throughout the year, she constructed a more complex approach to teaching much the same as her students were constructing mathematics for understanding.

Because of the success of this experiment, Cobb et al. (1990, 1991) gave workshops to other teachers to help them learn the teaching strategies of the experimental classroom. In the summer before the third year of their research, Cobb et al. conducted inservice workshops with 24 teachers in the same school. During this workshop, the researchers helped the teachers to see problems that might exist in their present instruction. The researchers helped teachers see problematic situations mainly by watching some of the students from the experimental classroom during problem-solving activities and assessment interviews.

The teachers began to question their beliefs about the ways children learn mathematics, especially the belief that correct answers imply conceptual understanding. Much of the time in the workshop was devoted to discussions about alternative instructional approaches: cooperative groups problem solving, discourse, and the use of manipulative materials in their instruction.

To add to the literature on constructivist learning, I review the work of Kamii. Kamii (1985, 1989) developed an approach to teaching that is almost exclusively learner-centered. Kamii used knowledge of how children construct knowledge much like the CGI researchers. She spent years interviewing children to understand their development of knowledge about place value, addition, and subtraction. Drawing on Piaget (1973), she claimed children actively construct knowledge of the world by continually interacting and adapting to their environment. Piaget (1973) and Kamii stated that in order to explain how students construct knowledge it is necessary to understand three different kinds of knowledge: physical knowledge, logico-mathematical knowledge, and social knowledge. Physical knowledge is knowledge about the qualities of objects--color, shape, weight. Unlike physical knowledge, logico-mathematical knowledge is not directly available from the external environment. The source of logico-mathematical knowledge is the child's mind and how the child interprets the relationships between qualities of objects. For instance, a child may identify blocks or shapes that are different or the same. The child constructs relationships about similarities and differences from within through his own mental activity. These relationships cannot be transmitted. Social knowledge comes from cultural conventions and is transmitted by oral communication. Examples of social knowledge are names such an eight or quarter.

According to Kamii (1989), children develop logico-mathematical knowledge through their own natural ability to think. Other people are not the sources of logico-mathematical knowledge. For example, to add 5 and 4, students have to think (1 + 1+ 1 + 1 + 1) + (1 + 1 + 1+ 1). Kamii (1990) stated that "this operation requires pupils to make two wholes (5 and 4) in their heads and then to make a higher-order whole (9) in which the original wholes (5 and 4) become parts" (p. 36). Addition involves thinking about relationships between numbers (e.g., 5 + 4 is one more than 4 + 4).

Using assumptions from Piaget's theory of learning, Kamii began a study in 1984. She worked with a public school where first- and second-graders did not have traditional experiences in learning mathematics. The students did not do paper and pencil algorithms. They learned place value, addition, and subtraction through group games and discussions. The teachers did not use textbooks; instead, they took advantage of the lives of the students to create problems for the children to solve. The principles of teaching that Kamii (1985) asked the teachers to use were (a) to encourage children to invent their own procedures rather than showing them how to solve problems, (b) to encourage children to invent many different ways of solving the same problem, (c) to refrain from reinforcing correct answers and correcting wrong ones, (d) to encourage the exchange of points of view among children, and (e) to encourage children to think rather than to write down answers.

Teachers wrote students' solutions to problems on the chalkboard to facilitate an exchange of viewpoints Teachers used daily situations of taking attendance and collecting lunch money as times to discuss mathematics. The teachers taught mathematics all through the day and not at a particular math period. The students never had a workbook or worksheet drill in mathematics. Kamii (1989) believed that drill on worksheets encourages the mechanical writing of answers and dependence on the teacher to know if an answer is correct. Children in the second grade learned to think about regrouping in addition and subtraction by constructing higher-order relationships in their heads based on their knowledge of adding and subtracting ones. They did not use manipulatives.

The teacher encouraged the children to agree or disagree during group problem solving. They decided among themselves whether answers were incorrect or correct. The teachers' role was to help children to clarify their answers. Instead of showing the students the right answer, the teacher asked for other students' solutions. The teacher listened and allowed the students to decide if the solutions were correct or incorrect.

Kamii (1989) followed the students' learning over a 5-year period. Kamii evaluated them by administering achievement tests (SAT), individual interviews, and a

Math Sampler test which she developed. On the Math Sampler she asked students questions involving estimation, mental arithmetic, and word problems. She designed the word problems so students could use their natural intelligence to solve them. On the Math Sampler, instead of choosing from several possible answers like on the achievement tests, children wrote their computations and answers in big blank spaces.

Kamii (1989) compared the achievement test scores of the students in the school where she was doing her research with students' scores who were taught in a comparable traditional school. Her findings showed that the *constructivist* students and the traditionally instructed students performed similarly on the SAT. However, during interviews in which Kamii asked students in both schools to solve problems, students in the constructivist group showed greater depth of understanding of place value, addition, and subtraction. On the interviews, there was a significant difference between the two groups ($p<.001$). Kamii's students had 67% correct responses while the traditional students achieved only 15%. On the Math Sampler, students in the constructivist learning group made fewer errors in reasoning.

From her study, Kamii (1989) concluded that the role of the teacher is to focus on children's thinking rather than on having children write correct answers. The teacher should encourage children to agree and disagree among themselves not reinforce right answers and correct wrong ones. Kamii said, "All the teacher can do is to stimulate the child's own construction of logico-mathematical knowledge from the inside" (p. 184).

Both the research of Kamii (1985, 1989) and that of Cobb et al. (1989, 1990, 1991) suggested that if student learning about mathematical understanding is to improve, teachers must teach in ways that facilitate students while they actively construct their own knowledge. Teachers must reconsider their beliefs about how children learn mathematics and how mathematics should be taught. In order to do this, teachers must see and try alternatives to the transmission approach that they experienced in their own pasts. These researchers believe that students must not only negotiate and share their thinking while creating knowledge, but also must discover and invent arithmetic.

Classroom Gestalt Paradigm

As a final approach of teaching, I review the research of Magdalene Lampert. Lampert (1990) combined learner-centered and teacher-centered paradigms to study teaching and learning in elementary classrooms. She also placed a heavy emphasis on mathematics content in her research. Unlike the Cobb et al. (1989, 1990, 1991) and Kamii (1985, 1989) studies, Lampert did not pose problems for problem solving but used problems to lead students to think about the mathematical operations as a basis of understanding relationships in mathematics. In her research, Lampert (1989, 1990, 1991, 1992) emphasized the nature of mathematics as a subject matter. Lampert demonstrated her belief in the value of studying teaching guided by what mathematics is and how mathematicians learn it. Lampert designed mathematics lessons by posing problems that will help students construct mathematical knowledge in the same ways that mathematicians discover new knowledge. Whereas the CGI, Kamii (1985, 1989), and Cobb et al. (1989, 1990, 1991) studies were heavily influenced by the cognitive developmental theory of Piaget, Lampert's research recycled ideas that date back to a time in mathematics education called the "new math" era when the nature of mathematics and its structure were important. She relied heavily on social interaction advocated by Vygotsky (1978).

Lampert (1990) combined action research (Ross, 1987) and sociolinguistic methods to study learning and teaching and stated that the purpose of action research "is not to derive new theories that can be applied to reform practice, but to subject theory to the conditions of practice . . . in a concrete situation so that theory and practice develop interactively" (Lampert, 1990, p. 37).

As part of 6 years of research as a professor at Michigan State University, Lampert began teaching mathematics to fourth- and fifth-grade students in a public school. During 6 months of this period, she audiotaped and videotaped lessons. Observers came three times per week to take field notes. Lampert kept all students' work and detailed field notes on the lessons. She collected data that included lesson and unit plans. The purpose of her research was to examine whether it was possible to make knowing mathematics in the classroom more like knowing mathematics in the discipline.

Lakatos (1976) explained that knowing mathematics in the discipline can be understood by realizing that mathematicians create mathematics through a process called "conscious guessing" about proofs. New knowledge is established by presenting a conjecture, arguing, and refuting about the evidence of the new conjecture, and deciding upon a proof of the conjecture. Mathematicians agree and disagree about the proofs until "truth" is established. Lampert (1990, 1991) believed this is different from how truth is established in school. In the traditional teaching of mathematics in school, truth is determined when the teacher says it is correct and knowing mathematics means remembering and applying the correct rules.

In order for students to come to know mathematics in ways of the discipline, Lampert (1989, 1990) noted that the teachers must alter their traditional role of being the authority for correct answers. Lampert (1989) said the teacher becomes the expert in the group who supports the students' construction of mathematical procedures and principles by choosing and posing the problems and developing the tools for the classroom communication or discourse (Vygotsky's zone of proximal development). Lampert's (1989, 1990, 1991) studies revealed how the teacher conveys to her students the demands of the mathematics subject matter with the challenge of helping students construct their own meaningful conceptual understandings. Consequently, the teacher is centrally important in helping students to construct an understanding of mathematics.

In Lampert's classes the students learned by participating in the solutions of problems that she posed. The content of the lessons was the students' arguments that supported or rejected the solution strategies (Lampert, 1990, 1991). The problems were rich in that they had alternate solutions and several ways to explain these solutions.

Students did not solve them by applying algorithms. Lampert (1989,1990) used familiar problems and tools, such as ones involving money, to get students to talk about their reasoning. From this reasoning, she then led them into unfamiliar territory where they constructed mathematical principles. Students found their own ways to adapt or invent new ways of using the tools.

With her research, Lampert (1986, 1989, 1990, 1991, 1992) concluded that students have the potential to gather information, to organize it, to conjecture, and to demonstrate and judge their solutions. Lampert (1986) stated that students need to be asked questions that can be "figured out" by thinking about the quantities and the symbols. Students should not be allowed to rely on memorized rules for moving numbers around. They should be challenged to make sense of their answers and see connections between in-school and out-of-school mathematics. Students should learn computations efficiently and effectively but only in connection to the meanings of the concepts underlying the computations. Lampert (1990) claimed that helping the students to know mathematics is like "teaching someone to dance, it require(s) some telling, some showing, and some doing it with them along with regular rehearsals" (p. 58). In Lampert's class students understand mathematics through a process of interaction with the "expert" of the group.

Lampert (1986, 1989, 1990, 1991, 1992) combined teacher and learner-centered paradigms with a focus on mathematics content to show us how effective teachers use their knowledge of mathematics and mathematics learning to promote an environment in which students communicate about mathematics like mathematicians. She demonstrated how teachers can help students establish rich connections between formal and informal knowledge, the familiar and the unfamiliar, and the abstract and the concrete (Vygotsky, 1962). Lampert used problems to lead students to think about the mathematical operations as a basis of understanding relationships in mathematics.

Summary

As a result of an increase in knowledge of how children learn mathematics, late in the 1980s, the National Council of Teachers of Mathematics began developing the *Curriculum and Evaluation Standards for School Mathematics.* In 1989 the *Standards* were published. These *Standards* not only emphasized learning through mathematical problem solving but also through mathematical reasoning and communication. They pointed out that the needs of society and how children learn mathematics should be addressed while children learn mathematical patterns and relationships. *The Standards* stated that with the advent of affordable calculators and computers, our society does not need students who can perform long and tedious computations. We need people who can think critically and be an informed electorate in our democratic society.

In their document, the NCTM (1989) affirmed their position that students should learn mathematics by constructing knowledge. In order for them to construct meaningful knowledge, however, students must be involved in *doing* mathematics, not just *knowing* mathematics. Students do mathematics as they "gather, discover, or create knowledge in the course of some activity having a purpose" (p. 7).

At one end of the continuum of research on elementary mathematics teaching we see the active teaching model of Good et al. (1983) with instruction using behaviorist learning theory--presenting small steps of the procedures for students to practice. At the other end of the continuum is Kamii (1985, 1989) and her emphasis on constructivist learning theory--allowing students to naturally construct mathematical knowledge. Finally, Lampert (1986, 1989, 1990, 1991, 1992) combined strong roles for both teacher and learner--posing the problems and guiding the students to construct the structure and theory of mathematics.

From this review we have learned that the research integrating elementary mathematics teaching and learning has evolved from just examining teacher actions to examining how teachers think and the effects of this thinking on their actions. We have learned that student thinking has affected teacher thinking and, in turn, has affected student performance. We have learned that students construct personal knowledge through social interaction in the presence of a knowledgeable teacher. This evolution of research has given us a clearer understanding of the complex interaction of teaching and learning in the classroom.

Research studies such as that of Good et al. (1983) are useful in building understanding of effective teaching but are not adequate for studying the complicated art of teaching. Leinhardt's (1986, 1988, 1989) research emphasized the importance of teacher planning. Teachers who help students understand have rich agendas of what students should be learning from the lesson. Leinhardt's expert-novice research and Hiebert and Wearne's (1985, 1986, 1991) content model of prescribing lessons based on content are important contributions because their research revealed how teacher knowledge can make a difference in teacher effectiveness.

Hiebert and Wearne (1985, 1986, 1991) integrated teaching and learning. These researchers provided teachers with information about how children learn in order to prescribe instruction that helps students learn specific content. In addition, research on Cognitively Guided Instruction has played a significant role in increasing understanding about how children learn content. Carpenter and Fennema (1991), Carpenter et al. (1989), and Fennema et al. (1989, 1991) provided teachers with knowledge that enables students to have the opportunities to learn to construct their own methods of solving problems rather than learning the teacher's method. Carpenter and Fennema's research gave teachers the knowledge of how children learn so that they can use this knowledge in making decisions about what and how to teach.

Cobb et al. (1989, 1990, 1991, 1992) and Kamii (1985, 1989) helped us understand how individual children construct knowledge through social interaction in the classroom. Students in Kamii's classrooms invent procedures by drawing on their own knowledge because they have no algorithms that they have learned to fall back on. Cobb et al. and Kamii assumed that learning mathematics happens in the social context of the classroom. They utilized Piaget's cognitive learning theory and Vygotsky's theory of social interaction. These researchers used everyday problem-solving situations to help children invent the procedures for finding solutions.

Lampert's (1986, 1989, 1990, 1991, 1992) research linked all the models--active teaching, teacher thinking, student thinking, and teaching with a constructivist epistemology. She is an *active teacher* who differs from the Good et al. (1983) traditional model of active teaching. Lampert does not state the purpose of the lesson or provide an outline of the lesson for the students as recommenced in the active teaching model. Lampert does not organize information into a series of small easy steps. She allows students to help create the direction of the lesson and the content. She acts as a guide for the students.

Implications for This Research

The underlying assumption of *The Standards* is that learning mathematics is a constructive process. When students use prior knowledge to construct new mathematical knowledge, they are learning in ways that help them *understand* mathematics and gain mathematical power. Teaching that emphasizes constructivist learning theory implies fundamental changes in the mathematics that will be taught and learned in elementary schools. Research in real elementary schools with real teachers can help educators and teachers to understand where to make the changes.

Lampert's (1986, 1989, 1990, 1991, 1992) teaching provided a model for me to use to look for real-life elementary teachers--active teachers using constructivist learning principles. Cognitively Guided Instruction helps provide models for real-life teachers who use knowledge about how children learn mathematical content. Hiebert and Wearne (1985, 1986, 1991) helped me understand how to find teachers who help students connect procedural knowledge and conceptual knowledge to build relational understanding. Cobb et al. (1989, 1990, 1991, 1992) and Kamii (1985, 1989) gave guidance for me in finding teachers who help children construct knowledge through social interaction.

Investigating real-life teachers can help answer questions about how teachers who believe in constructivist learning principles use their knowledge and beliefs to make decisions and provide instruction. How does a teacher support students in developing their ideas? How does a teacher decide if an answer is mathematically valid or invalid? How can teachers understand the mathematics constructed from a child's point of view? Unless teachers have seen or read about alternate approaches to teaching, they will teach content in mathematics similarly to how they were taught (Ball, 1988). By examining the practice of teaching in actual classrooms, researchers can provide different models of what teaching "looks like" when students are constructing their own knowledge of mathematics. As a theory, constructivism provides a powerful basis for learning. However, we lack knowledge of what this theory looks like in practice. My research provides us with the opportunity to see what constructivist approaches to teaching looks like in practice.

CHAPTER III
RESEARCH METHODOLOGY AND PROCEDURES

In this chapter, I describe my research methods and procedures. I include a discussion of the research model, the theoretical perspective of the research, research methods and procedures, the selection of participants, the data collection and analysis, my qualifications and biases, and the issue of reliability and validity.

With my research methods, I answer questions that are specific to one fourth-grade teacher. I profile her classroom teaching and present her instructional decisions. These descriptions reveal the teacher's goals, thinking, and decision making as she constructs her own meaning of constructivist teaching. My research methods help me describe the dilemmas this teacher encounters and, in some cases, resolves as she provides opportunities for students to reason and communicate in order to gain the mathematical power.

Research Model

I used ethnographic research methods to investigate this elementary teacher while she taught mathematics. Ethnographic methods permitted me to understand the life of the teacher within "the framework of the culture" (Bogdan & Biklen, 1982, p. 37). This qualitative methodology is composed of several characteristics (Bogdan & Biklen, 1982; Reichardt & Cook, 1979): (a) The research is conducted within the particular setting under study; (b) the researcher is the main research instrument; (c) the data are descriptive; (d) the researcher obtains meaning from the context and from the participants' perspectives; (e) the data are analyzed inductively; (f) the research process is more important than obtained products; and (g) a holistic picture is created. A brief explanation of each characteristic follows.

Research Is Conducted Within the Particular Setting Under Study

I became an insider, a researcher who aimed to look at what was happening in the classroom through the eyes of the teacher. Owens (1982) stated that "qualitative inquiry seeks to understand human behavior and human experience from the actor's own frame of reference, not the frame of reference of the investigator" (p. 7). I used the natural setting of the classroom to conduct my observation of the teacher's classroom instruction and the classroom interaction. After school, I interviewed the teacher in the classroom.

Researcher Is the Main Research Instrument

During the mathematics lessons, I observed and took notes. I videotaped and audiotaped what was happening in the classroom. While students worked during class, I observed what the students were doing. During interviews with the teacher, I took notes and audiotaped while the teacher answered my questions. I prepared questions that help me understand her thinking and why she made certain decisions. I prepared questions to help me evaluate her knowledge and understand her beliefs about mathematics and mathematics teaching and learning. I planned questions that encouraged the teacher to reflect on why she chose particular teaching methods or assessments. I asked her to evaluate specific examples of students' work. I asked her how and why she might teach differently if she had to teach a lesson again. I asked the teacher what she thought about the students' learning.

Being a part of the mathematics classroom helped me understand how the teacher taught for understanding of mathematics and how the students reacted to her teaching. Rather than using a quantitative instrument that might only reveal numbers that fit into statistical models, I was the instrument of research who evaluated what was going on in the classroom. I analyzed the data in terms of answering my research questions.

Data Are Descriptive

In the following chapters, I provide thick description of the setting in order to communicate the meanings of how the teacher defined constructivist teaching. The term, *thick description*, used by Geertz (1973) means to provide a description that moves the reader to the context with all the "insight, understanding, and illumination not only of the facts or the events in the case but also of the texture, the quality, and the power of the context as the participants in the situation experienced it" (Owens, 1982, p. 8). These thick descriptions help present the teacher's instructional techniques and behavior while she helped her students construct mathematical knowledge. I describe the learning environment that is created during the interactions among the teacher and her students.

Researcher Obtains Meaning From the Context and From the Participants' Perspectives

Smith and Heshusius (1986) stated that qualitative research "presents a description constrained by values and interests to be compared with other descriptions constrained by other values and interests" (p. 11). I must consider that each description is based on a specific context that is a situated perspective. A situation is not a single event happening in isolation but an experience that is peculiarly connected to the real world. Dewey (1938) noted that a situation is "an *individual* situation, indivisible and unduplicable" (p. 68). I tried to understand what kinds of knowledge that the teacher had about the content that she was teaching, how and why she believed students should learn certain mathematical understandings, and why she believed that certain teaching strategies were good for students to achieve this understanding. By seeking to understand the teacher's perspective or conceptions, I understood more clearly her reasons for decisions and classroom teaching. I reveal with the descriptions in these writings how the teacher believes children learn mathematics. I describe her reasons behind practice. I write, in her own words, why she chooses certain problems, manipulative materials, assessments, and seating arrangements for students. I present the data to explain the dilemmas that she thinks about as she provides her students with the opportunities to gain mathematical power.

Data Are Analyzed Inductively

Through comprehensive day-to-day observations and interviews with the teacher, I acquired data that helped me understand what the teacher actually knows and believes about mathematics and mathematics teaching and learning. I asked her to review videos of her teaching and learning episodes from the previous days' lessons to help me understand the *intuitive screen* (Liston & Zeichner, 1991) through which she saw classroom events. I analyzed data from the interviews. From the observations on a daily basis, I made conclusions and thought of questions. I tested this analysis through asking more interview questions and observing more lessons.

Research Process Is More Important Than Obtained Products

Goetz and LeCompte (1984) stated that "data collection and analysis are inextricably linked in ethnography because the ethnographer may not know what questions to ask until impressions and perceptions have been analyzed and tentative conclusions have been formulated" (p. 165). Spradley (1980) discussed the importance of process when analyzing data. He used the term *hypothesis-oriented* ethnography. My inquiry was hypothesis oriented because I selected the site and collected data based on my hypotheses that the teacher's classroom practice might prove to portray constructivist learning principles. This hypothesis provided the stimulus for observations and interviews.

From my work with inservice and preservice teachers, I had seen differences in teacher knowledge and beliefs about mathematics and mathematics teaching and learning. From my reading of *The Standards* (NCTM, 1989, 1991) and from my own teaching experience, I knew that teachers must, to some point, believe in constructivist views of learning in order for them to implement the teaching practices called for in both *Standards* documents. Even though the teacher in my study had a commitment to constructivist theory, she could have possible differences in interpretations of what constructivist learning involves. I probed constantly with observations and with interview questions to discover her interpretations. For instance, I asked, "What do you think it means to understand mathematics?" "How do you help students come to correct answers?" "How do you assess individual knowledge within whole-group classroom interaction?"

After I hypothesized that the teacher's conception of mathematics and how it should be taught were, indeed, in agreement with constructivism, I proceeded to collect data on my research question, *what happens when an elementary teacher implements constructivist teaching and learning of mathematics in elementary classrooms?*

In addition, I gathered data that helped me answer the more specific research questions that I gave in the first chapter. After I began collecting data, I formed more questions to ask the teacher and to guide my observations.

1. What are the teacher's instructional decisions in the following areas:
    a. lesson planning?
    b. classroom discourse?
    c. assessment of teaching and learning?
2. What does the classroom teaching look like in the following areas:
    a. lesson planning?
    b. classroom discourse?
    c. assessment of teaching and learning?
3. What dilemmas of teaching mathematics does the teacher encounter during instructional decision making and classroom practice?
4. How does the teacher define constructivist teaching and mathematics learning related to
    a. an examination of beliefs
    b. an examination of knowledge
5. What are the connections between the teacher's definitions of constructivist learning and her instructional decisions and classroom practice?

Holistic Picture Is Created

I attempted to arrive at a holistic picture of constructivist teaching in practice. I looked at the physical layout of the classrooms, the teacher's background and experience, the number of students in each class, the teaching materials, the classroom interaction---all were valuable for me in understanding the teacher's decision making and behavior. I tried to interpret her beliefs and knowledge by not only asking her questions but also observing her practice. I found that the I could better understand her view about mathematics teaching and learning by observing what she did in the classroom setting.

Bush (1982) presented the following assumptions about ethnographic methodology:

> human behavior is complexity influenced by the context in which it occurs, and the meaning of observed human behavior an be understood only with respect to the perspectives of the participants in the setting. Strategies in ethnographic research have been viewed as phenomenological, naturalistic, holistic, and multimodal. It is phenomenological in that it represents the world view of the participants in the study. It is naturalistic in that it involves accounts of phenomena as they occur in realistic contexts. It is holistic in that descriptions of total phenomena in their human behavior and beliefs. It is multimodal in that a variety of research techniques can be used. (p. 9)

Theoretical Perspective

I assumed a theoretical perspective of symbolic interactionism for this study. According to this perspective, "objects are social products in that they are formed and transformed by the defining process that takes place in social interaction" (Blumer, 1969, p. 69). Blumer's perspective of social interaction coincides with constructivist learning theory. According to this view, mathematics is the product of the interactions among teachers and students in classrooms. Blumer characterized symbolic interaction as a process of how a person interprets and explains the fundamentals of the situation. To understand how the teacher interpreted and understood the fundamentals of the interactions in the mathematics classroom, I entered the context of the classroom environment and sought to experience the process by which she made decisions. Observing the social interactions among teacher and students and understanding her thinking gave me the opportunity to understand her perspectives and practices.

Blumer (1967) suggested a qualitative, naturalistic approach as a compatible methodology with the symbolic-interactionism perspective and investigating the perspectives of others. He established the naturalistic method as the approach for studying symbolic interactionism. According to Blumer (1969), “no theorizing, however ingenious, and no observance of scientific protocol, however meticulous, are substitutes for developing a familiarity with what is actually going on in the spheres of life under study” (p. 39). The ethnographic research approach of participant observation, interviewing, and collecting artifacts provided a compatible methodology for investigating what happens when a teacher implements constructivist teaching and learning of mathematics in an elementary classroom.

Selection of the Participants and Site Description

I observed in 10 different elementary classrooms. I identified five potential teachers by their responses on a *Mathematics Beliefs Scales* questionnaire developed by Fennema, Carpenter, and Peterson (1987) (Appendix A). The teachers with the lowest scores had the strongest beliefs about constructivist learning and teaching. Most questions related to whether the teacher believed that children construct their own mathematical knowledge or if children need to first be shown how to solve problems and tasks. After giving and scoring the questionnaire and with more observations in the classrooms, I evaluated the consistency of their answers on the questionnaire to their classroom practice. The observations helped me to find those teachers who were implementing *The Standards* while helping their students gain mathematical power. Because I saw inconsistencies between answers on the questionnaire to actual teaching practice, I eliminated one of the five teachers. I then interviewed the remaining four teachers and discussed my study in general terms. The interviews helped me identify the teachers’ levels of interest in participating in my study. I also wanted the teachers to enjoy talking about their ideas and to believe that my research would be valuable.

In the first month, I observed two of the teachers every school day for 4 weeks. By the second month, I observed the remaining two teachers every school day for 4 weeks. By the end of second month, I eliminated two more teachers. I finally chose two of the teachers for two reasons: (a) They believed in the constructivist theory of learning and teaching in mathematics, and (b) they had begun implementing the approach to teaching described in both *Standards* documents. I continued to observe the two remaining teachers every school day for the next 2 1/2 months. I collected and analyzed the data from both teachers. However, I finally chose to write this investigation about only one of the two teachers because I became excited and interested in her approach to teaching. I decided that I could present a more clear description and portrayal of constructivist learning by using one powerful teacher.

Over the course of the study, I collected my data in four self-contained, intermediate-level, elementary mathematics classrooms located in a small suburban county near a large southeastern city. I observed 125 hours of teaching and conducted 14 interviews. I received permission from the teachers, the principal, the school administration, and the University of Florida Institutional Review Board.

Research Methods and Procedures

Through the ethnographic method Spradley (1979, 1980) called the Developmental Research Sequence, I investigated how this teacher interpreted and implemented constructivist learning of mathematics. Spradley developed this research sequence as a methodology for investigating *meaning*. I adapted his model for studying the meaning behind the teacher's instructional decisions and practice.

The goals of ethnographic inquiry are both descriptive and analytic. With this type of inquiry the researcher seeks to understand the cultural meanings which organize and direct the lives of individuals within a social setting (Spradley, 1979, 1980). The goals and methods applied in the classroom are basically the same as those applied in the study of any social setting. Researchers who use participant observation methods for studying classroom cultures seek to illuminate the meanings that the individuals make of their roles. I attempted to understand the meaning that this fourth-grade teacher made of what it means to be a "constructivist teacher."

Ethnographic methodology is cyclic rather than linear. The researcher begins a cycle of questioning, collecting data, recording data, and analyzing data. Then, the process begins again with questioning, collecting, recording, and analyzing data. The researcher begins with very broad questions, observes in the setting to answer these questions, analyzes the data from the observations, creates new more focused questions, and begins the cycle again. Throughout the time of the study, I repeated the sequence of questioning, collecting, recording, and analyzing.

Data Collection

During my investigation, I collected three types of data--observation data, interview data, and documents. These three types of data helped me learn the culture of this classroom by making inferences. According to Spradley (1980), researchers make these inferences by using three types of information. He stated that "we observe what people do (cultural behavior), we observe things people make and use (cultural artifacts), and we listen to what people say (speech messages)" (p. 10). In this study, I observed the teachers' behavior while teaching mathematics, the use of instructional materials, and the paper work prepared by the teacher (worksheets, assessments). I collected some students' work. Finally, I observed the teacher's and students' discourse about mathematics. I collected data in the words of the informants using field notes, videotapes, and audiotapes.

Goetz and LeCompte (1984) recommended multiple methods of collecting data; they refer to what Denzin (1978) called "triangulation" for enhancing the credibility of research findings. Denzin said that "triangulation is a plan of action that will raise sociologists above the personalistic biases that stem from single methodologies. By combining methods . . . observers can partially overcome the deficiencies that flow from one investigator or one method" (p. 294). The three main methods of data collection that I used are participant observation, informant interviewing, and collecting artifacts.

Participation Observation

Becker and Geer (1967) defined participant observation as

> that method in which the observer participates in the daily life of the people under study, either openly in the role of researcher or covertly in some disguised role, observing things that happen, listening to what is said, and questioning people, over some length of time. (p. 109)

My level of actual participation in this classroom was minimal. I characterize my participation as passive (Spradley, 1980). Spradley described "passive participation" of a researcher as "the ethnographer engaged in passive participation is present at the scene of action but does not participate or interact with other people to any great extent" (p. 59). The teacher introduced me as "Mrs. Steele, someone who is here to gather some information about teaching mathematics." The teacher explained that I was not there to help students with their work, but only to observe. To my knowledge, no one told them additional information about me. During class, I sought to have little interaction with the students or the teacher. Occasionally, I walked around the room to observe the students during individual and group activities. No students asked me for help.

I directed my observations toward understanding how the teacher constructed her own meaning of constructivist teaching. I focused on the interactions among the teacher and students. I observed the discourse during lessons. In the *Professional Standards for Teaching Mathematics* (NCTM, 1991), discourse is defined as "the ways of representing, thinking, talking, and agreeing and disagreeing that teachers and students use to engage in . . . tasks" (p. 45). I observed Who talked? To whom? About what? In what ways? I examined the teacher's role, the students' role, and the role of materials used to represent or model the mathematics content during the discourse. I examined the opportunities that the teacher gave the students to learn and construct mathematics. Some of the questions that I asked myself follow: "Whose ideas and ways of knowing are accepted and whose are not?" "What makes an answer true?" "How does this teacher interpret what the students are saying?" I analyzed the learning environment created by the teacher as she interacted with students. I asked, "Does the teacher respect students' ideas?" "What kinds of thinking does she encourage?" I observed and asked questions about how the teacher evaluated the students. How did she observe, listen, and gather information about students to assess what they were learning? How did the teacher assess the students' conceptual knowledge?

I spent most of the time in an unobtrusive position at the back of the room, observing, listening, videotaping, and taking field notes. I use the term "field notes" to mean descriptions that I wrote about the teacher and students, about the objects they used, about lessons, and about problems and activities. I tried to use the words of the informants when I took notes about the interactions during class, rather than translating what the participants said. I audiotaped verbatim accounts of the words of the teachers and students. When describing observations, I used concrete language instead of summarizing and generalizing. For instance, I wrote down observations, such as Marsha waved her hand to get the teacher's attention and shouted, "I know the answer."

In the margins of the field notes, I included ideas and questions that occurred to me. Although I wrote, on the spot, possible interpretations in the margins, most interpretations were written up later during data analysis. I took field notes in a narrative format. I did not use a formal field-note observation format. As I collected data, I looked for emerging patterns and themes that I had found during the data analysis. These patterns and themes helped direct my future observations (Bogdan & Biklen, 1982).

I videotaped, spaced over time, 16 hours of mathematics lessons taught by the teacher. At first, I allowed the teacher to choose the lessons. However, after the teacher became comfortable being videotaped, she allowed me to tape any lessons that I chose. After viewing and reflecting on these videos, I listed questions that I wanted to ask of the teacher. I chose segments of the videos to transcribe into field-note protocols. I made comments in the margins of the protocols. Bogdan and Biklen (1982) stated that "observer's comments are sections of the field notes in which the researcher records his or her own thoughts and feelings" (p. 149). The comments allowed me to hypothesize and make conjectures about the data. They also included any original or different ideas I thought of in understanding a particularly uncertain or unclear observation. The teacher and I viewed two videotapes together, and I allowed her to view some videos at home. During the viewing, I asked the teacher to talk or write about and explain her reasons for decisions that she made while teaching. I asked her to discuss her thinking about the students during the lesson.

Interviewing

I used two types of interviewing, formal interviewing and informal interviewing (Spradley, 1980). Spradley explained that "an informal ethnographic interview occurs whenever you ask someone a question during the course of participant observation" (p. 123). During a slow time or before or after class, I asked the teacher to clarify some classroom event or episode. For example, when she decided to choose fraction circles instead of pattern blocks to teach mixed numbers, I asked her, "Why did you make this choice?" I asked questions to get information about the teacher's reactions to and reflections on the mathematics class. Examples included the following: "Did the lesson go as you thought it would?" "Why did you change the wording of the problem?" However, at all times, I avoided disrupting classroom activities when asking questions of the teacher during instructional time.

Spradley (1980) described a formal interview as one that "usually occurs at an appointed time and results from a specific request to hold the interview" (p. 124). Spradley added that formal interviews can provide valuable information that is not available from informal conversations. I collected data from six formal interviews with this teacher. I developed most of the questions for the interviews after analyzing my field notes from observations (Appendix B). I had developed some structured questions, however, before I began my data collection. These previously developed questions helped me to understand the teacher's knowledge and beliefs about mathematics and mathematics teaching and learning. I asked many questions to get at the teacher's thinking about instruction and reasons for making certain decisions. Before two videotaped lessons, I gave the teacher some "pre-observational questions" (Appendix C) about her planning for that lesson. I asked her to write the answers and return them to me before she taught the lesson. While reading her answers, I developed questions of my own that guided my observations for this lesson. After these two lessons, I asked her write answers to "post-observation" questions (Appendix D). I audiotaped all of the interviews. I transcribed all audiotaped interviews to help me analyze the data. All interviews took place in the classroom after school. There were minimal disturbances. Most of them lasted between 1 and 1 1/2 hours.

I conducted the interviews using the "general interview guide approach" (Patton, 1980, p. 197). With the general interview guide approach, I listed questions or issues that I wanted to explore. Many times when the teacher answered a question, this answer would lead me to another question. Over time, however, she answered all my questions. I probed and followed up responses to questions as I thought it was appropriate. I probed for the reasons behind her classroom practice. My questions came from several sources: questions from *Cognitively Guided Instruction* on beliefs about how children learn mathematics, questions from the *Mathematics Beliefs Scales* questionnaire (Fennema et al., 1987), and questions that I created to determine the teacher's definition of constructivist teaching and learning and her thinking about instruction. I asked the teacher several questions concerning the dilemmas that she had encountered while teaching mathematics in the ways that she had. If I observed dilemmas, I asked her questions to ascertain if she also considered these situations dilemmas.

Sometimes during the interviews, the teacher and I viewed videotapes of the lessons. I developed some interview questions to obtain information from the teacher while she observed certain teaching episodes on the videotapes. I started and stopped the tapes while I ask her to explain her thinking and behavior. These questions helped me clarify the teacher's decisions about practice during the lessons. They helped me to understand her interactive thoughts about the content and the students.

Some interview questions focused on how she planned her lessons. Examples of questions included the following: "Do you plan a lesson around a certain learning objective?" "How do you plan how long to devote to each concept?" "What will you do if a student cannot solve the problem?"

Collecting Artifacts

To supplement the field notes that I took during participant observations, videotapes, and interviews, I collected or viewed various artifacts from the teachers' classrooms, such as copies of pages in teachers' planbooks, students' work assignments, students' portfolios, and students' tests. Goetz and LeCompte (1984) noted that "such artifacts provide evidence for the topics and questions ethnographers [researchers] address because they are material manifestations of the beliefs and behaviors that constitute a culture" (p. 153). My goal was to overcome any shortcomings of participant observation and interviewing by triangulating the three methods. Combining methods is recommended by many qualitative researchers (Becker & Geer, 1969; Denzin, 1978; Goetz & LeCompte, 1984).

Data Analysis

Data collection and data analysis were fundamentally linked to my research approach based on the theoretical perspective of symbolic interactionism and constructivist learning theory. I chose a qualitative, naturalistic methodology because it was compatible with the social-interactionist viewpoint of constructivists who believe that students' constructions of knowledge are facilitated through social interaction. Because in this study my purpose was to describe and analyze one elementary school teacher's approach to implementing *The Standards*, accordingly, I chose a method of data analysis that was compatible with the qualitative methodology of ethnography. Goetz and LeCompte (1981) stated that "the general purpose of the research, the nature of the research problem or question, and the theoretical perspectives that inform the research problem and intrigues the researcher" (p. 64) should influence the choice of analysis process.

I wanted to understand the complexity and the reality of teaching fourth-grade mathematics from a constructivist learning perspective. To achieve this purpose, I gathered a large amount and variety of data. Given the amount and variety of data that I collected over a 4-and-1/2 month period, it was critical that I began to interpret and analyze the data from the first day of observation. Bogdan and Biklen (1982) noted that the researcher needs to conduct a systematic examination of the data by "working with data, organizing it, breaking it into manageable units, synthesizing it, searching for patterns, discovering what is important and what is to be learned, and deciding what you will tell others" (p. 145).

I followed Spradley's (1980) method of data analysis to "determine its parts, the relationship among parts and their relationship to the whole" (p. 85). As soon as possible after I collected field notes and transcribed the interviews, I analyzed the data for emerging patterns of ideas and activities. I combined and compared new data with previously collected data. I used this information to guide my future observations and interviews. Through simultaneous data collection and analysis, I began to form and test hypotheses concerning the teacher's practice and her instructional decisions. I looked for patterns in her practice, patterns in her planning, patterns in her assessment, and patterns in the ways she made decisions. After I formed a hypothesis, such as the teacher builds instruction on students' prior knowledge, I tried to find indications that would agree with or contradict this hypothesis. In this way, I revised my questions and conjectured about patterns repeatedly over the course of the study.

Throughout the analysis, during and after the data collection, I used Spradley's (1980) model to direct my search for patterns. The first phase of analysis involved identifying categories of meaning or what Spradley calls "cultural domains" (p. 87). To identify domains, I spent time reading and analyzing the protocols with specific questions in mind. Spradley suggested kinds of questions such as the following: Are there *kinds* of things here? Are there *ways to do things* here? Are there *reasons* for doing things? Some of my domains were "kinds of problems," "ways of assessing students," and "types of manipulatives used." After I had identified a large number of domains, I began to focus on the ones that appeared to be the most often repeated in order to guide the remainder of my study. Throughout the data collection and analysis, I read and domained the information into categories in order to organize the data. I used words and phrases to represent patterns and domains. I developed files that contained parts of transcripts and protocols that represented different domains to give a clearer view of what data was available. I reread the data and added new data to change existing domains, get rid of old ones, or add new ones. I compared the transcripts from the interviews with transcripts of fieldnotes of classroom observations. I used the audiotapes and videotapes to help me to review data and further investigate emerging patterns.

Another part of the sequence in Spradley's process of data analysis is componential analysis. This analysis is a search for the characteristics or components of each domain. Spradley said that the researcher must find these components by digging deeper into the culture. An example of how I used componential analysis to find subsets that related to domains as wholes was with the domain titled "ways of assessing students." I broke this domain into its subsets--ways of assessing during cooperative learning, ways of assessing during whole group discourse, and ways of assessing during individual work.

By using this analytic process of ethnographic research, I moved from asking general questions about the study to asking more specific questions. These questions directed my subsequent observations. I looked for relationships between components of domains to reduce the number of domains.

I compared and cross-checked the data to help me search for themes. Spradley (1980) stated that "themes are assertions that have a high degree of generality. They apply to numerous situations and recur in two or more domain" (p. 141). A theme ties together the patterns of the pieces in the setting. I searched for themes of the meanings that recurred across the domains. I tried to unify the themes. Glaser and Strauss (1967) called this search for unity a need to become "theoretically saturated" (p. 111). Saturation means that new pieces of data do not point to new aspects of the domain. Again, I formed hypotheses and tested them by asking the teacher if she recognized these themes about her teaching.

Even though the themes might have been at a level of knowledge that the teacher had not voiced, she could recognize underlying themes that described her classrooms. When the teacher recognized the themes, I had evidence that confirmed the hypotheses. My discovery of themes helped explain the teacher's practice and the reasons behind practical decisions. My discoveries also helped me to understand and explain the teacher's thinking and goals.

Researcher Qualifications and Biases

In a study like this one, it was essential that personal biases that might affect collection, interpretation, analysis, and results of data be brought into the open. Having spent all but the first 5 years of my life in contact with some form of schooling--as a student, as a teacher, and as a teacher educator--I began my study with my own conceptions about mathematics and about mathematics teaching and learning. These conceptions influenced my choice of purpose and methodology for the study. I know they also influenced my focus during classroom observations. I worked, however, to maintain my role of researcher, as opposed to one of teacher.

I worked to view a familiar situation through the eyes of a stranger. Although I could not forget or ignore what I knew or believed, I took steps to use my knowledge of mathematics and constructivist beliefs about mathematics teaching and learning as an advantage in the research process. I attempted to understand the experience from the point of view of the participants. I tried to assume that I understood nothing about teaching mathematics in elementary school and needed to learn everything that I could. As I compared and contrasted the data with my own experiences as a teacher, I understand some things I had not previously understood. For instance, I understood more about how to interpret students' solution strategies.

Even though my personal background could have affected findings of the study, my qualifications have made me uniquely able to answer my own research questions. Since in qualitative research the researcher is the principal instrument, my qualifications are considered relevant. I have experience in collecting and analyzing qualitative data. Wolcott (1976) said that the most important qualifications for ethnographic research is experience conducting fieldwork. Another qualification that researchers consider important is writing up the findings of qualitative data. I have experience in writing up findings of qualitative studies that I have conducted. Merriam (1985) wrote that the sensitivity and integrity of the investigator during the data gathering and analysis are important in writing and sharing the findings.

I have had the following experiences that relate to Wolcott's (1976) standards for qualitative research:

1. I have earned a B.A. in elementary education and a B.S. in mathematics.
2. I was a classroom teacher for 13 years, 10 years at the elementary level and 3 years at the secondary level. At the secondary level I taught mathematics from general mathematics to calculus.
3. I have earned an M.A. in instructional communication.
4. I have completed coursework for a Ph.D. in curriculum and instruction, including courses in elementary curriculum and mathematics education.
5. I have taken three courses that have provided a theoretical and practical background in qualitative research. One course was in qualitative research foundations, and the other two were methods of ethnographic research.
6. I have completed two ethnographic studies--one in a seventh-grade algebra classroom and one in an elementary mathematics methods course. I have written a paper for each study and presented them at the American Educational Research Association annual meetings.
7. I have been a college instructor in mathematics and mathematics education for 7 years and have supervised student teachers for 1 year.
8. I have worked on improving my writing skills by preparing papers for conferences and manuscripts for publication.

As a researcher, I confronted and sought to eliminate any bias that would affect the findings of my study. Wolcott (1976) suggested that the researcher "needs to grapple with his own underlying assumptions' and . . . recognize the kinds of evidence he is most attracted to in building his account" (p. 27). Kirk and Miller (1986) recommended that researchers reflect on and examine their own values. By listing relevant beliefs, Ross (1978) suggested that the researcher presents awareness of them and, thus, provides the reader with a foundation for evaluating the study.

1. I believe that children construct their own knowledge of mathematics rather than acting as passive "receivers" of knowledge. I believe that children know mathematics by *doing* mathematics, not computing mathematics.

2. I believe that the teacher's role in teaching mathematics is to guide the student in constructing relational understanding of mathematics.

3. I believe that teachers should teach mathematics to children by focusing on how children learn mathematical concepts.

4. I believe that mathematics should be taught within a heterogeneous-grouped class of students.

5. I believe that classrooms are complex environments where teachers and students interact and influence each other's thinking and behavior.

Validity and Reliability

Goetz and LeCompte (1984) stated, "credibility mandates that canons of reliability and validity be addressed wherever ethnographic techniques are used" (p. 210). They added that ignoring threats to credibility weakens research and that ethnographers must consider lack of reliability and validity to be serious threats to the value of their results. Validity has to do with "how one's finding match reality" (Merriam, 1988, p. 166). Merriam said that reliability deals with "the extent to which one's finding can be replicated" (p. 170). Merriam suggested that validity and reliability depends upon the researcher's presentation of "people's constructions of reality properly" (p. 168).

Some of the steps that I took to ensure the reliability and validity of my study's findings have been suggested by Goetz and LeCompte (1984) and have already been discussed. For instance, I spent a long period of time collecting data so that I could become familiar with the classroom, students, and the teacher. In addition, I used several methods of data collection so that I could compare and contrast the patterns of meanings. I sought to keep my biases in check. I cross-checked data from interviews with data from observations to try to identify any inconsistencies. I reviewed my findings with the participant to see if they agreed. I have established "a chain of evidence" (Yin, 1982, p. 91) that will reveal to the reader how I drew my conclusions.

It is important that my study is of use to other researchers. Consequently, I tried to achieve some external validity or generalizability of the findings. Eisner (1981) said that generalizability is possible in qualitative research because of the belief that the "general resides in the particular" (p. 7). Yin (1989) suggested that analytic generalization does not rely on samples and populations. He also stated that qualitative research may generalize a particular set of results to a broader theory. I extend my findings to the constructivist learning theory.

Stake (1978) advocated the idea of "naturalistic generalization" by saying that "the demands for typicality and representiveness yield to needs for assurance that the target . . . is properly described. As readers recognize similarities . . . of interest to them, they establish the basis for naturalistic generalization" (p. 7).

Stake believed that the written presentation of the research should be descriptive, comprehensive, and complete. He also said that qualitative studies must be well written. Through the detail of the presentation of my data, I make it possible for the reader to establish the credibility and reasonableness of my findings. Therefore, I view my research as "successful" because I comprehensively and completely describe the relationship between the teacher's constructivist thinking of mathematics teaching and learning and her practice of mathematics teaching and learning to help students gain mathematical power.

Summary

Constructivist teaching and learning has been critical in recent research in mathematics education and has provided a basis for recent reform efforts like those advocated in *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) and *The Professional Standards for Teaching Mathematics* (NCTM, 1991). Although constructivist learning theory of mathematics is potentially critical to making changes in mathematics teaching, there is not presently a model of how this theory is reflected in a teacher's thinking and behavior. Models of teaching based on constructivism are needed. The purpose of this dissertation is to provide a possible model. In the following discussion, I will profile one elementary teacher who is interpreting and implementing constructivist learning theory while teaching mathematics. I will describe and present information about the instructional decisions and strategies that are being used by this teacher who is constructing her own meaning of teaching for students to construct personal mathematical knowledge. I will discuss the role of teacher as she negotiates with students to find and adapt activities for them to construct mathematical knowledge. I will present and describe dilemmas and struggles that this teacher encounters when making instructional decisions. Data will be presented from whole-class, constructivist teaching episodes where the interactions among teacher and students will be shown to affect opportunities for their students to learn mathematical power.

CHAPTER IV
A PORTRAYAL OF TEACHING IN BARBARA CLARK'S CLASSROOM

Introduction

In this chapter, you will meet a fourth-grade teacher. Through the teaching episodes in her classroom, you will get a general picture of her mathematics teaching. This general picture will include the context of this fourth-grade, the teacher's background, and the components of her mathematics teaching. You will read about a typical class day, a day in February, and see the teacher encourage students to construct mathematics independently and collectively.

My initial problem was to find a teacher who taught mathematics based on constructivist principals. I asked several administrators, teachers, and parents in a southern school district to suggest an elementary mathematics teacher whom they believed to be effective. The same name kept coming up--Barbara Clark. I went to Mustang Elementary to meet Mrs. Clark. I observed her mathematics teaching on two different occasions. I was excited by what I saw. Mrs. Clark, I saw, questioned children and often asked them to explain their thinking. The children were actively involved in doing mathematics. They manipulated materials, drew diagrams, and worked cooperatively to solve problems. In short, I saw a teacher who helped students construct their own knowledge.

I asked Mrs. Clark to complete the *Mathematics Beliefs Scales* questionnaire (Fennema, Carpenter, & Peterson, 1987) so that I could assess her beliefs about how children learn mathematics, about how mathematics should be taught, and about the relationship between learning concepts and procedures. Her score on the questionnaire confirmed what I had observed. I had found my teacher.

The Context

Mrs. Clark taught in a self-contained, fourth-grade, public school classroom in a suburban community adjacent to a large southern city. The school system has demonstrated its commitment to change mathematics and science teaching. In the years prior to my study, the district had offered five inservice workshops to help teachers improve their mathematics and science instruction. The school system is a county system that has 12 elementary schools with levels K-6. There are 12,800 students attending the elementary schools and 450 elementary teachers. Mustang Elementary is located in a lower-middle to middle-middle social and economic area--one-third of the students receive free or reduced lunch. At Mustang Elementary there are 54 teachers and 1,268 students. There are 5 fourth-grade teachers and 136 fourth-grade students. Mrs. Clark has 25 students and teaches all the elementary subjects. Seventeen of the students are Caucasian; eight of the children are from minorities--3 African-American, 3 Hispanic, and 2 Asian-American.

The classroom is in one of 30 portable buildings at Mustang Elementary. The only classes housed within the main brick structure are the kindergarten and first-grade classes. This classroom has three large book shelves, two filing cabinets, and three large double-door cabinets filled with books, resources, and manipulatives. Children's work is taped on walls and windows, hanging from the ceiling, and attached to bulletin boards. One long chalkboard extends three-fourths of the length of the front of the classroom. At the left of this chalkboard is a mathematics bulletin board that Mrs. Clark uses everyday when she teaches mathematics. When I first observed in the classroom, the students' desks were placed in groups of five. The desks and chairs were separate. Later, the teacher moved the desks to other positions--a circle, a horseshoe shape, or straight rows. Students often worked in pairs. Mrs. Clark was usually talking with a student or group of students. In the 4 months I was there, she never sat down during mathematics class except to help a student.

It became apparent on the first day that Mrs. Clark's major aim was to help students learn to think about mathematics. With the problems and activities that she chose and her interaction with and questioning of the students, she had established a community in which the children and she could build a classroom of mathematics thinkers. The students and the teacher had constructed, over time, a dialectic learning environment through everyday interactions. She encouraged the students to reason about mathematics, not learn it mechanically. The first time I interviewed her, she said,

> I've been right up front with them that what I'm looking for is their thinking. What are they thinking about and where are they coming from? I want the students to become better thinkers in all situations. I think these experiences help them to think better in other classes, not just better in math. From these experiences, they become more comfortable, much more excited about math. I think if you require students to really get into [a] narrow path of thinking, you don't allow them to let their own thoughts come into it and their own reasoning come into it. I think they need to reason things out for themselves.

Even though Mrs. Clark did not call her teaching an approach based on constructivist learning theory, she was teaching in ways consistent with the active learning characterized within this theory. The children were learning mathematics by constructing their own knowledge rather than depending upon the knowledge of the teacher's.

The Teacher

Mrs. Clark has been a teacher for all her adult life. She has taught for 23 years--17 years at the elementary level and 6 at the junior high level. She has a Bachelor of Science degree in elementary education with a certification in middle-school mathematics. She is a member of the National Council of Teachers of Mathematics, and she sits on the board of her state's Council of Teachers of Mathematics. Mrs. Clark is the "lead teacher" for the fourth-grade team in her school. She has been her school's representative to county mathematics teacher enhancement training and the district mathematics council. Mrs. Clark brings back to the teachers what she has learned in workshops. She is involved in changing mathematics instruction in elementary schools in her county. She helps plan mathematics field days and has worked on the mathematics curriculum writing team for her county. Mrs. Clark conducts "family math" seminars and problem-solving activity workshops for other teachers and for parents in her county. She gives an open house at the beginning of the year to explain her mathematics teaching to parents of her students. Many residents of the county regard Mrs. Clark as one the best elementary mathematics teachers in her county. Her endeavors are confirmed by her students' CTBS test scores.

She has sat on state and national panel discussions concerning preparing teachers of elementary and middle schools. She is the co-founder and first president of a support group for teachers who are implementing the active learning principles suggested by the NCTM *Standards* documents. She has an excellent reputation at community, state, and national levels for teaching excellence. She was given the Presidential Award for Excellence in Mathematics Teaching. Mrs. Clark is constantly attempting to improve her knowledge and upgrade her skills in mathematics teaching.

Mrs. Clark enjoys teaching mathematics and is extremely interested in teaching mathematics. She readily admits that it is her favorite subject and yearns for students to become involved in the problems or tasks. Her love of teaching mathematics is evident everyday. She says, "It gets you going when you see that something is really working. That just *fuels your fire*." She demonstrates extraordinary knowledge and ability in building a learning community of mathematical thinkers.

The Lessons

Let me describe a typical day in Mrs. Clark's mathematics class. Mrs. Clark begins every day teaching mathematics. Most of her mathematics classes have six sections. Students know they will work in these six sections: journal writing, daily calendar, counting tape, daily deposit, *The Standards*, and problem of the day. Mathematics class begins at 7:45 with journal writing and usually ends by 9:00. She is usually not in a hurry and concentrates on a few really significant problems for students to investigate. She says her goal is very simple. "I want my students to have daily experiences with mathematics that are both meaningful and thought provoking." The structure of the mathematics class might include cooperative-group problem-solving situations, hands-on activities, or whole-class interaction.

Math Journals

During the first 15 or 20 minutes of school, students are busily writing in their "math logs," the name that Mrs. Clark has given to the math journals that each student must keep throughout the year. In these journals students keep their answers and solution strategies to problems. They began writing at 7:45 and work until about 8:00. Most students are interested in writing their ideas or solutions to the problems or tasks that Mrs. Clark has written on the chalkboard. They want to be able to share their thinking. If the problems are challenging, students know they can use this time to think about the strategies they use to solve the problems. Sometimes they write a paragraph about how they solved the problem. Sometimes they draw a picture or a diagram to explain their thinking. They also might explain how they used a calculator to solve the problems. They know, however, that they cannot present an answer without explaining how they found it. As Mrs. Clark says, "They know they must explain how they solved the problem."

The writing in their math logs helps the students focus on each problem and reflect about their own thinking. By thinking about working out their solutions before the actual discussion begins, the students can think about the questions they know Mrs. Clark will be asking. They learn to evaluate their own approaches and to begin to clarify their thinking. By working in their math logs, the students stimulate their prior knowledge of the mathematical concepts.

During this writing time, the students usually work independently and do not ask questions of Mrs. Clark. They spend their time in thinking and writing in their logs. Even though Mrs. Clark expresses delight when students came up with solutions to problems, they do not rush to show her their answers to check to see if they are correct. They do not work quickly through their problems to see who can get the answers first. Students know that her approval comes from being able to explain and justify their reasoning, not in speed or accuracy of their answers. While they work on or after they complete the morning's problems or activities, some students read, get drinks, go to the restroom, and generally prepare for the day.

Mrs. Clark is usually working on attendance, lunch, and other opening-of-the-day tasks while the students work in their logs. While she does these tasks, however, she is constantly observing the students. When she has completed these tasks, she spends about 5 minutes walking around the room to examine some of the students' ideas or solutions.

Mrs. Clark uses the journal writing to assess students' prior knowledge and ideas about the problems for that day. She observes students as they are writing, and she uses the students' comments and strategies to begin class discussions.

The Calendar

Usually by 8:00, "calendar" time begins. It is the first of three sections of the mathematics class that are constant but progressive in thinking difficulty as the school year moves ahead. In this section, Mrs. Clark uses a display of a calendar page that shows the days of the month (see sample calendar in Figure 4). The calendar is a large part of a bulletin board on the left side at the front of the room. Each day of the week is represented by a colored geometric shape. Each day they add a new shape. The calendar helps the students create patterns involving shape, size, and color. There are numbered calendar geometric shapes to create patterns that develop throughout the month. Usually, Mrs. Clark plans the pattern for the month ahead of time. However, sometimes the students form the patterns as the days of the month progress. For example, there might be a pattern that is a red triangle, red square, blue triangle, blue square, red triangle, where a star might appear on each multiple of 3. By using the calendar shapes to develop patterns, Mrs. Clark asks students to search for connections between the sizes, the colors, and the shapes that might reveal the pattern layout of the calendar for the month.

The students examine the past shapes, colors, and numbers to make predictions on the day's possible geometric cutout. At the first of the month, the students can only guess

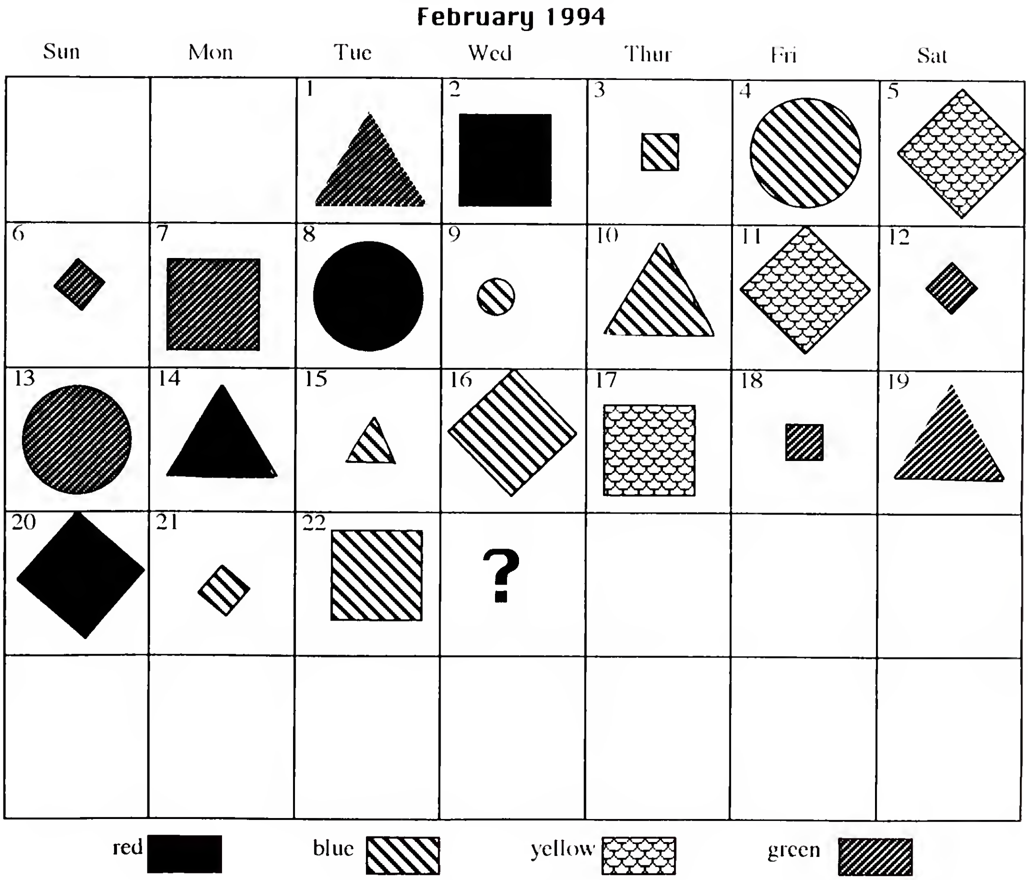


Figure 4. February's Calendar

because there are not many shapes, but gradually as the month proceeds, they develop theories about the patterns. They learn to analyze and continue patterns, see patterns of odd and even numbers, and reason logically. They find patterns by observing what has happened in previous days and extend the patterns into the future. An added benefit is that they learn to recognize and name types of triangles and other polygons. Mrs. Clark explains what she wants the students to learn from the calendar experiences,

> I want them to look for patterns. The obvious pattern is why you got the next cutout, but they start looking for all kinds of patterns--vertical, horizontal, diagonal. I want them to hear and try to use correct vocabulary. They learn the shapes. The calendar helps them with mathematical concepts and skills that relate to things we might study later. I want to help create some prior knowledge for later.

The teachers' focus for the month of February is on the attributes that will encourage logical thinking. Mrs. Clark uses the pattern pieces of circles, squares, diamonds, and triangles to form a geometric pattern in which each pattern piece differs from the one previous to it by two attributes, either by shape, color, or size (see Figure 4). Each possible piece is like the one previous to it by one attribute. The possible colors are yellow, red, blue, or green. The choices for size are large and small. For example, if Wednesday is a large red triangle, then Tuesday can be a small blue triangle because it differs on two attributes, size and shape, but is alike on one attribute, the shape of both are triangles. Each day of the month the students predict what will be that day's geometric shape. Since Mrs. Clark focuses on process more than product, the children's thinking and learning guides her morning mathematics class. The following excerpt from the class discussion is for February 23rd.

> Mrs. Clark has asked the students for their predictions about today's possible pattern piece. The teacher writes their predictions on the chalkboard.

L	Y	T	(Large Yellow Triangle)
L	Y	C	(Large Yellow Circle)
L	Y	D	(Large Yellow Diamond)
L	Y	S	(Large Yellow Square)
L	G	C	(Large Green Circle)

Mrs. Clark: There are a lot of predictions for large. Joan, why did you say large?

Joan: Yesterday we had a large and the one before that was large, the one before that was small, and before that was large, and before that was large. So you have large, large, small, large, large, small, large. (She is looking at the last 7 days.)

Mrs. Clark: Joan went back and the pattern is showing us large, large

Students: Small, large, large, small.

Mrs. Clark: She went backwards and she says we only have one large and the day before was small, so today would have to be a large. Anybody want to make an argument for a small? (Mrs. Clark waits. No responses.) No. Ann, do you have a different argument for large?

Ann: All of the diameters (she means diagonals) have stayed the same size. Large stays on this diameter.

Mrs. Clark: (corrects Ann) Diagonal. So Ann is saying that all of the diagonals have stayed the same. So she is thinking that this diagonal ought to stay the same size and that would be large. Any other arguments for large? Okay. Are we all in agreement on large?

Students: Yes.

Notice that Mrs. Clark does not just ask for a prediction but asks students to support their prediction with reasoning. She guides her students in understanding that mathematical thinking is based on logical reasoning. She encourages them to give their predictions for size of the geometric cutout. After they have agreed upon the size, she takes them to the next attribute.

Mrs. Clark puts a check above the column that has L (large).

Mrs. Clark: Now, the color. We have a lot of thoughts that it might be yellow. Anybody have an argument on why it should be yellow. William?

William: Following the forward diagonal, I think what should be next is yellow (see Saturday in Figure 4).

5
11
17
?

Mrs. Clark: I like what you said, "Following the forward diagonal." (Mrs. Clark goes over to the calendar and points out the forward diagonal.)

Mrs. Clark: Somebody else said, yellow, tell me why you picked it. Sharon?

Sharon: On the diagonal there is yellow, yellow, yellow. Then it should have another yellow.

Mrs. Clark: So Sharon is saying that all the diagonals have the same color. Anyone else want to give their reason? Eddie?

Eddie: We add 6 to get the yellow. So 17 + 6 would be 23. Today is the 23rd, so it should be yellow.

Mrs. Clark: A different reason for yellow?

Hal: If you look at the vertical pattern (He is pointing across with his hand.), no, horizontal

Mrs. Clark: There you go.

Hal: You see green, red, blue, blue, yellow. I see a red, blue, blue, and today would be yellow.

Mrs. Clark: So he's looking at the horizontal direction, as we go along (pointing on the calendar) we have green, red, blue, blue and he is saying yellow.

Mrs. Clark: Another reason, Hal?

Hal: If you go down, there is red, then you have blue, blue, so it would have to be yellow.

Mrs. Clark: So, looking at the vertical direction, (pointing on the calendar at the vertical pattern) we also have red, blue, blue, yellow. Here's a blue, blue, and yellow (goes to a different vertical column on the calendar). Here's a blue, blue, and here's what after?

Students: Yellow.

Mrs. Clark: So, we have looked at all directions now. We have looked at the horizontal direction, at the vertical direction, and at the diagonal direction. Every direction that we have looked, we have got what?

Students: Yellow.

Mrs. Clark: So is yellow okay?

Students: Yes.

Mrs. Clark puts a check over the column that has Y (yellow).

Most students have very clear reasons for choosing the color. They have come to understand many patterns on the calendar about the attribute of color. As the month proceeds, they have heard some of these arguments for colors before and have remembered them. Next, Mrs. Clark wants to emphasize the rule that they have developed for this month that one attribute stays the same and two change. She guides the students to clarify and synthesize what they have already predicted.

Mrs. Clark: Oh, now, we have to look at the shape. The problem that we always have. Before we discuss it [the shape], let's go back and figure out which attribute stayed the same. Is large the same or different?

Students: Same. (Mrs. Clark writes S (same) over the column for size.)

Mrs. Clark: Is yellow the same or different?

Students: Different. (Mrs. Clark writes D (different) over the column for color.)

Mrs. Clark: Different. What does that mean about the shape?

Hal: Different.

Mrs. Clark: Different. Because only one attribute stays the same. So the shape is going to be different. We have decided that today what is going to stay the same is the size. Right? That means the other two attributes are going to be different. Which one of those choices can we take out? Cathy?

Mrs. Clark reminds the students that they have already reached a conclusion that the only attribute that stays the same is the size. She tries to help the students keep an eye on where they are headed.

Cathy: Large yellow square. (Mrs. Clark marks out L Y S.)

Mrs. Clark: Tell me why you said square.

Cathy: Because yesterday was a square. So today wouldn't be.

Mrs. Clark: Yesterday was a square and we said the shape has to . . . ?

Students: Change.

Mrs. Clark: Or be different today. That leaves us with three possible choices for our shape. Anybody want to make an argument for diamond? Why should diamond be the choice. Jim?

Jim: Diamond, diamond, square, diamond, diamond, square on the diagonal pattern.

Mrs. Clark: Okay, the diagonal pattern has been diamond, diamond, square. You think that it is going to repeat with a diamond. Okay, that's an argument for diamond. Anybody have a different argument for diamond? (No responses) I'll put a check there to show that we had an argument for that. Okay? How about circle? Ken?

Ken: You have diamond, diamond, square, circle. Starting with Saturday, you had diamond, diamond, square, so today would be circle. (This day is Monday.)

Mrs. Clark: Ken sees a pattern for the past days [points to 11, 12, 13]. He gave an argument for circle. Another argument for circle?

Sharon: We haven't had a circle yet this week.

Mrs. Clark: Oh. We haven't had one this week. Did we have one last week?

Students' mathematical arguments continue until a case had been built for all three shapes (diamond, circle, triangle). Students conclude the lesson by voting on the shape that each decides had the best argument. Mrs. Clark writes the number of votes for each shape on the chalkboard. The vote is about even among all three shapes. Mrs. Clark presents the calendar piece for the day.

Mrs. Clark: It must be a----triangle.

Students: Yeah! Yeah! Ah! Ah!

Mrs. Clark: Let's hope the month doesn't end before we figure out this shape pattern. So today is a triangle. It *is* large and yellow.

As you can see from the previous dialog, Mrs. Clark validates each student's thinking by encouraging them to express their thinking and calling each student by name. She compliments William on his use of the term *forward diagonal.* She helps them to develop their thinking and express their thinking using mathematical terms. She calls attention to the language they use. For instance, she helps Ann to use the word diagonal instead of diameter and encourages Hal to correct vertical to horizontal.

The students give their own predictions and describe why they think they know the shape, color, and size. They share the various ways that they arrived at their answers. The students seem to have succeeded with the pattern for the color and size. However, as their arguments for the shape demonstrates, they are having trouble predicting the shape. By the end of the month, they still had not gotten the pattern for the shape. Mrs. Clark did not explain the pattern to them. She said to them, “If I told you, than I wouldn’t allow you to think. I like to hear what your arguments are about what it is going to be.”

Many times students will change their minds after a student has given an especially convincing argument for a shape, color, or size. For example, in the previous calendar lesson Pat had said that he thought that the day’s geometric shape would be a large, green, circle. After several students had given their reasons for choosing yellow, Mrs. Clark asked Pat to tell his argument for green.

Mrs. Clark: Pat, do you want to tell us your argument for green?

Pat: Because of yellow, green, green, yellow, green, green, yellow, (He is pointing to the right side of the calendar. He sounds uncertain and then stops.)

Mrs. Clark: You decided that you didn’t like green? (Pat shakes his head) No. We’ll mark that one out then. (Mrs. Clark puts an X through L G C.)

The students have the opportunity to talk about what they see and to understand and learn from one another. They discover that there are many approaches to finding answers. At the end of the month, the students describe all the different patterns they have found for that month--large, large, small; red, blue, blue; circle, diamond, diamond, square. They can be said to communicate mathematically--triangle, diagonal, vertical, pattern.

During an interview Mrs. Clark talks about the day’s (February 23rd) discussion of the calendar pattern.

> This [pattern] is a little bit iffy. They’ve gotten the large and the small down. Last month there were two attributes that were the same and one was different. That one was easier for them. I’ve gotten them to key in on those that changed. That was a hard thing, especially with this one at the beginning--because there is so much to

> consider. Size is easy. Color is easier. But predicting the shape is still a problem. That's one thing I like about this pattern. There is an easier one, a middle one, and a harder one for them to deal with. So, it gives everyone success. It they were all three still hard to predict and come up with, they would be frustrated by now. The calendar is a little scary for me sometimes because they can come up with some patterns I don't even think of.

Counting Tape

The second part of this day's mathematics class is the counting tape. The counting tape is an unfolding number line that helps students begin to identify the patterns and order of our numbers. It is a creative way of learning and applying the multiplication facts and divisibility rules for factors two through nine. The students begin to learn number theory, the relationships among numbers. They examine theory of numbers through factors or multiples, divisibility rules, and prime numbers.

The counting tape is a paper strip for keeping track of the number of days in school. Each school day, beginning with the first day of school, a new number is added to the counting tape. Examining this daily number helps students discover the relationships between multiples of numbers. Students look for multiples for each new number. Mrs. Clark uses cutouts to symbolize each multiple from two through nine. She uses cutouts as symbols of the concept of number rather than using the numerals as symbols. She wants students to learn that numbers have a value; therefore, she does not use numerals as symbols to represent the concept of number. They learn that numbers can be represented in different ways. Mrs. Clarks attaches these symbols to the number line under the numbers of which they are factors (Figures 5 and 6).

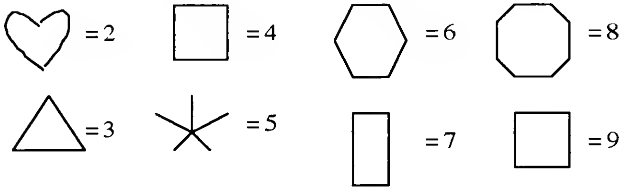


Figure 5. Symbols for numbers

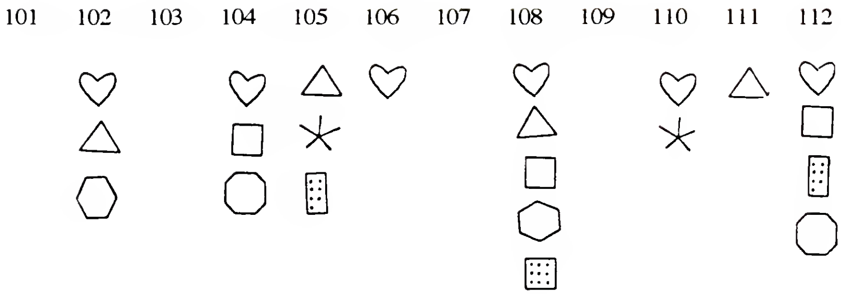


Figure 6. Sample counting tape

Mrs. Clark might ask, "If you did not know every multiple of 8, how could you tell which numbers would be multiples of 8 just by looking at the counting tape?" John answers, "All the multiples of 8 have hearts and squares and octagons." Or Mrs. Clark asks the students what do the multiples of 5 have in common. While students respond by saying they all end in 0 or 5, Mrs. Clark makes a list of these on the chalkboard--5, 10, 15, . . . 90, 95, 100, 105, 110.

The following vignette is an example of the discourse about the counting tape. This is day 112 of the school year. The students are deciding which are the factors of this number.

Mrs. Clark: Is 2 a factor of 112? And why? Allen?

Allen: Yes. 112 is an even number.

Mrs. Clark: He said, "Yes," because 112 is an even number. He says even because it ends in

Students: 0, 2, 4, 6, 8.

Mrs.Clark: Is 3 a factor of 112 and why or why not? George?

George: I added the numbers and got 4. Three is not a factor because the sum is not 3, 6, 9. (They have learned the divisibility rule for deciding if 3 is a factor of a number. The rule is that the digits of the number must add up to 3, 6, or 9.)

Mrs.Clark: He added the digits 1 + 1 + 2 and got what?

Students: 4.

Mrs. Clark: Four as the sum. He said 3 is not a factor because the sum is not 3, 6, or 9.

Hal: (Hal is waving his hand.) We had a 3 yesterday.

Mrs. Clark: Okay. Another reason is because we had a 3 yesterday. Is 4 a factor of 112? William?

William: Yes.

Mrs.Clark: Tell me why.

William: We had 108 and a 4. One hundred eight, then you add 4 and 108 is 112.

Mrs.Clark: On day 108, we had a 4. And 108 + 4 = 112. That is a reason why. Can anybody give me another reason? Dan?

Dan: If 4 goes into last two digits.

Mrs. Clark: If 4 goes into the last two digits then that number is divisible by 4. Does 4 go evenly into 12? (They haven't known about the rule for 4--4 must divide into the last two digits evenly--for long since they are only on day 112. On day 100 they first developed the rule when there were more than two digits. Many of the students do not know this rule yet.)

Students: Yes.

Mrs.Clark: What about 5? Emmy?

Emmy: No, the last digit of 112 is not a 0 or 5.

Mrs.Clark: She said it doesn't end in 0 or 5. How about 6? Valerie?

Valerie: No, we don't have a 2 and 3. (The divisibility rule for 6 to be a factor is that you must have both a 2 and a 3 as factors. Erin previously gave the reason why there was not a 3.)

Mrs.Clark: No, because we don't have a 2 and a 3. What if I say 2 *or* 3?

Jim: No, because you'd be saying if you had a 2, you'd have a 6. If you had a 3, you'd have a 6. If you say 2, it's just 2. It doesn't mean 3.

Mrs.Clark: Okay. There's is a difference between *and* and *or*. We need to look carefully at that. We only have 2. How about 7? Patty Ann?

By emphasizing the difference between *and* and *or*, Mrs. Clark is already preparing students to understand formal logic of "if . . . then" statements that they will encounter in their high school courses. This discussion continues:

Patty Ann: Yes, because if we had 7 on day 105. 105 + 7 is 112.

Mrs.Clark: (Mrs. Clark writes 105 + 7 on the chalkboard.) 105 + 7 is 112. She went back to the last multiple of 7 and added 7 more and got 112. Does anybody know a different way I can test if 112 is divisible by 7 without looking back at the previous multiple? Cathy?

Cathy: Divide it.

Mrs.Clark: Divide it. The divisibility rule for 7 is to

Students: Divide.

Mrs.Clark: Let's do that division. It helps us to look at how to do that [division] so that when we get to our division, we will have a little bit better idea of what we are doing. Seven into 11 goes how many times?

Mrs. Clark introduces naturally something (division algorithm) they'll learn formally later. This comes up regularly in her teaching. The students have not formally had the division algorithm in her class. Mrs. Clark has given the students opportunities to learn division with base-ten blocks and counters. However, she says that, in her experience, the students never connect the concept of division with the algorithm. So with the counting tape, after guiding them through developing the algorithm, they practice the procedure together. Sometimes she has them write it in the air with her. She says, "I am just helping them learn some prior knowledge so that when we get to division, they won't freeze and be afraid." Today, Mrs. Clark writes on the chalkboard as the students in unison tell her the steps for solving the algorithm. They divide, subtract, and bring down. She shows them how to decide correctly, where to put the numbers in the quotient, and how to write the remainder. They go on to complete the divisibility rules through multiples for nine.

Every day the students continue this type of discussion until they finish with the multiples for numbers 1 through 9. Some of the students easily voice their knowledge of the divisibility rules. They have not learned them all at once but have heard about each rule as it was needed. Mrs. Clark did not explicitly tell them the rule. She guided them in developing each new rule. She helped students learn to identify counting patterns for even and odd numbers, to identify patterns of numbers from 10 to 100, and to see how these patterns are repeated with higher numbers. They begin informal explorations of division with and without remainders and learn mental math. Mrs. Clark helps the students develop mental strategies for thinking so that they can make sense of the relationships between the numbers. Her goal is to help students recognize and apply number theory so they can plan approaches for using these thinking capabilities.

The students reinforce their multiplication and division knowledge gradually. By looking at the relationships, they construct their own divisibility rules which help them work mathematics mentally. Learning multiples is a way of gaining understanding of the meaning of the divisibility rules. Finally, the students learn the relationships between multiples and are able to predict when the next multiples of certain numbers will appear on the counting tape. These experiences help students move from only recognizing patterns to a higher level of using the patterns as a problem-solving strategy. Perhaps, more importantly, they begin to understand that mathematics is a science of patterns and order.

The teacher usually leads the counting tape section, but students also enjoy playing the “role of teacher. ” Some students seem to be good at understanding the counting tape. They volunteer to lead the discussion. They often copy Mrs. Clark’s questions or create their own techniques. Mrs. Clark says she likes the students to act as leaders in this and in other discussions because

> I think that when they are up there, they see things a little differently. I can get an idea of how they are or are not understanding things--whether they can handle the questions that the kids ask them. How they handle them. They are forced to use the vocabulary. They are forced to make decisions about whether students responses are right or wrong. Yet, they have my support. It forces them to communicate and forces the other students to communicate with them. Some

> children who won't raise their hand with me will raise their hand with a child up there. I don't ask them to be a leader unless they are comfortable with the topic. I don't force anybody. When they are comfortable with it and they are feeling good about it, then I like to get them up there to be the leader.

When Mrs. Clark allows the students to act as leaders, she evaluates their knowledge. She is also seeking to create a nonthreatening environment in which students are willing to take risks to explain their thinking.

Daily Deposit

The fourth section of mathematics class is the daily deposit. The purpose of the daily deposit is to help students learn place value concepts, develop number sense for large numbers, and understand multiplication and addition by making predictions. The students multiply the day of the month by $100 and add that amount to the deposit that began the first day of school. For example, on the fifth day of the month, the students put $500 in the bank (5 times $100). The students put the money into pockets that represent dollars in the bank. The pockets are stapled to the bulletin board in order of place value (see Figure 7). The place value section of the board has pockets for the ones, tens, hundreds, thousands, ten thousands, hundred thousands, and millions.

When putting money in the pockets, the students count out the previous day's dollars and decide how much regrouping they will need to do to get the new deposit. These types of activities help them learn how to count and add money. The activities improve students' number sense, specifically, the number sense of big numbers to one million. The students also learned rounding of numbers to all place values. Mrs. Clark

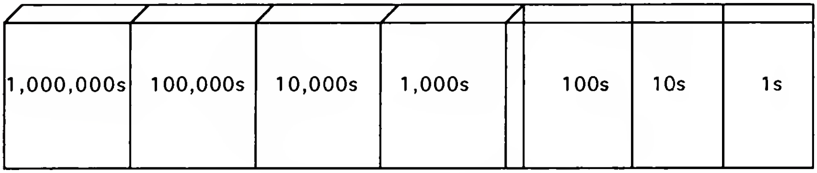


Figure 7. Daily deposit

made statements like, "Suppose you are rounding 8,000 to the nearest 10,000?" Or "In the bank there is $282,500. Round it to the nearest 10,000." Mrs. Clark gives students opportunities in this section of the mathematics class to understand and use numbers to compute and estimate.

By the time I began observing at the beginning of the second semester, students were adding the big numbers mentally. They were no longer considering the ones and tens place values but were only looking at the hundreds, thousands, ten thousands, hundred thousands, and millions.

As the money accumulates in the bank, the students get the opportunity for place value discussion. The following is a discussion about figuring out the daily deposit and the bank balance for Monday, March 7th.

Mrs. Clark: So how much do we have today?

Students: $1, 300.

Mrs. Clark: Raise your hand and tell me how you got $1, 300 to add. Emmy?

Emily: I had 600 and 700. (Sunday was the 6th and Monday was the 7th. So she added 6 + 7 =13. Then she multiplied by 100.)

Mrs. Clark: I have $300 in here [the pocket]. How many hundred should I add to it?

Emmy: 3.

Mrs. Clark: 3 what?

Emmy: 3 hundred.

Mrs. Clark: Do I need to regroup?

Students: No.

Mrs. Clark: No. Why not? Missy?

Missy: (no response)

Mrs. Clark: What is the total of 300 and 300?

Missy: 600.

Mrs. Clark: Do I have to regroup?

Missy: No.

Mrs. Clark: Why? How much do I need to regroup?

Missy: 1,000.

Mrs. Clark: I have to have a 1,000. I don't have a 1,000. So that gives me a total of how much in the hundreds place?

Students: 600.

They continue the discussion from the ten thousands through the millions. Notice that Mrs. Clark is very particular that they must say 1,000, 10,000, and later 100,000, not just 1 or 10 or 100. The total for today was \$266,600 (\$265,300 + \$1300 = \$266,600). After the discussion of today's amount, she asked students to predict whether they might need to regroup tomorrow.

Mrs. Clark often has the students act as leaders for this section of the mathematics time also. After the students feel comfortable with counting the money and understanding place value, leading the class in the discussion is a concrete, beneficial way for them to reinforce their learning. Many of the students enjoy leading the daily deposit because they like to count the money.

Perhaps the most beneficial part of the daily deposit is the prediction portion of the activity. The dilemma at the beginning of the year is predicting whether the bank will accumulate \$1,000,000 by the end of the year. All year long students keep this question in mind. Periodically, they predict whether they will reach \$1,000,000 by the end of the year. Mrs. Clark repeats and restates the question throughout the year. She encourages the students to be independent thinkers and find reliable ways to predict. She encourages them to estimate and to base their estimate on "something." Some students boldly guess. However, with experience they have found that just guessing is not useful. In the following vignette, some of the students are explaining why they think that they will not get to \$1,000,000.

Mrs. Clark: The other day, Hal made a comment that I didn't come back to. Hal's comment was, "We're not going to make it." How many agree with Hal that we might not make it? (Several hands are up.) How many think we are still going to make it? (A few hands are up.) Why do you think that we might not make it, Hal?

Hal: Because we still have almost $700,000 more to get to $1,000,000. And today's total--all we have right now is $266,600. Then you have got $700,000 more to get. And it has already been 108 days.

Mrs. Clark: Half of the school year is 90 days. We are past that. He is saying that we have already been in school 108 days.

Jim: We are not even half way there.

Mrs. Clark: What would half of a million be?

Hal: 5. Uh, 5 . . .

Mrs. Clark: Be careful how you say it. Ann?

Ann: 500,000.

Mrs. Clark: Say it again, Ann.

Ann: 500,000.

Jim: We are not even going to get to that.

Mrs. Clark: Okay. We are more than half way through with school. Hal is thinking that--and Jim and some of the others are agreeing with him--that we should be at least half way to a million with our money. They are saying that we are not far enough yet. Jim, why do you say that?

Jim: Because we only have $266,600 now and we still have to get more than 200, 000 more just to get to half. We have $700,000 to get to a million.

Mrs. Clark allows the students to rethink their predictions as the year progresses, and they hear other students' reasons for their predictions. After hearing the reasoning behind another's thinking, the students clarify their own thinking. After the explanations from Hal and Jim, Mrs. Clark asked, "If you changed your mind from last month, raise your hand. You have to tell your reasons. If you did not change your mind, I'd also like to hear your reasons." Mrs. Clark says that they have good arguments for their points of view and asks the class to vote. More than half of the class agree that they will not reach a million but will probably get close to 500,000. Mrs. Clark ends the discussion by saying, "I'm glad you brought up that point, Hal. That gave us the chance to discuss it."

By the end of the month, the amount of money added for that month gets very large. Mrs. Clark suggests many times that students use "mental math" to estimate the amount of money they have added. They discuss the answers to decide if they are reasonable and why. They check their answers for reasonableness with their calculators. They also practice rounding numbers so that they do not need to rely on calculators to check if answer is reasonable. Mrs. Clark says,

> Mental math is a goal I have for a lot of students. Some of them have gotten off the calculator. Some of them have realized it's more trouble to get out that calculator. That's good. That's what I want them to realize--you don't need a calculator to do everything.

The dialog from the predictions shows that students are beginning to develop a number sense about numbers through their mental math, estimation, and prediction experiences. Mrs. Clark works to help them develop number sense through understanding the relative magnitude of numbers and through developing mental flexibility with numbers. She helps them to connect large numbers to the real world. She works to make the students feel comfortable using large numbers and relate the numbers to context. Since helping students develop number sense intuition takes years and is a way of thinking that should pervade all aspects of mathematics teaching and learning, Mrs. Clark uses activities of this type everyday.

The Standards

The fifth segment of mathematics class represents some element of *The Standards* (1989). Mrs. Clark explains that she uses this segment and the others to develop the mathematical concepts emphasized in *The Standards. The Standards* emphasize helping children gain mathematical power through problem solving and mathematical reasoning. They recommend increased attention to geometry, measurement, number sense, estimation, and mental math. Mrs. Clark uses this section to help students learn these important topics.

She introduces concepts that the students have not yet studied and also introduces concepts that complement what they are presently studying. Mrs. Clark says,

> I try to get this part of the math class to complement what we are doing in the book and to generate some things for the next unit we are going to do--like fractions--I want them to have some ideas and some confidence. . . .

During my observations, some of the topics that she used in *The Standards* section were measuring length in different units and changing the units to fractions and mixed numbers, learning to compute adding and subtracting fractions with hands-on and problem-solving contexts, developing algorithms, learning geometric thinking, and understanding probability. Many of the problems were based on geometric concepts. Mrs. Clark thinks that geometric concepts are extremely important when helping students to develop spatial thinking, visualization, reasoning, and the logical thinking that will help them in their later learning.

The following problems are examples of some problems that Mrs. Clark used to incorporate suggestions of *The Standards.* Can more than one circle have the same center? How many different triangles do you see in the following figure? All squares are rhombuses or all rhombuses are squares? Does a square always have the same value for perimeter and area?

Today's lesson is about measurement, an area that *The Standards* recommend for emphasis. The problem is as follows:

> If you have a circle drawn on your paper, how can you find the center of the circle if you don't already know where it is?

The students are enjoying investigating this problem. They are anxious to get through the first four sections of mathematics class so they can talk about how they solved this problem. They are excited to share their solutions. Mrs. Clark asks Gerry to go up to the chalkboard and show how she found the center of the circle. Gerry takes a plastic lid, draws a circle, then uses a ruler to draw a diameter. Mrs. Clark asks her to tell what she is thinking as she draws. Gerry says, "Take the ruler and find a point that you think is the center and mark it. Then move the ruler around and mark it and keep doing it." Gerry measures from the point that she has marked as the center to the edge of the circle and gets 5 inches. She keeps drawing diameters (see Figure 8). The discussion continues,

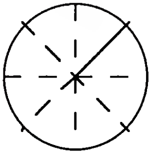


Figure 8. Gerry's Drawing

Mrs. Clark:	Why do you think that it was important to have 5 inches or how did you choose where you started?
Gerry:	Because it looks like it is about the middle.
Mrs. Clark:	She is picking a point. Guessing where the center might be. Looking at it and moving it around to see if it stays the same at 5. But look around here, Gerry, we have a problem. It's over 5 there. (Gerry tries to adjust for this by measuring again and checking the distance to the edge of the circle in several places.)
Mrs. Clark:	What's happening to the other end of your ruler, Gerry? Where your right hand is?
Gerry:	It's going out a little.
Mrs. Clark:	Is your left hand slipping? Is it hard to hold the same?
Gerry:	Yes.
Mrs. Clark:	Okay. We could do it that way. The problems we have are that we have to adjust and it is hard to hold the ruler in the same place as it goes around. If we worked at it we could probably find the center, couldn't we? Another suggestion. Ken? (Many students want to show their method.)
Ken:	I think it would be better if you found half of it.

Ken goes up to the chalkboard, picks up the ruler, and turns it around so that the end with 1 inch is on the edge of the circle. Mrs. Clark asked him why he changed ends. He says, "Because I had the 12 up there and I needed to have the 1." Ken gets 9 for the width of the circle. He says, "It's 9. So half of 9 is 4 1/2." The discussion continues, Mrs. Clark: Let's think about what Ken did. When he measured from one point on the edge of the circle to another point on the edge of the circle, what was he measuring for?

Students: Diameter. (Some students say the diagonal, but most say it correctly.)

Ken draws that diameter. Then he measures and draws another one (see Figure 9).

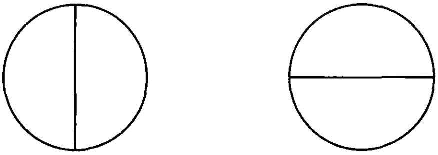


Figure 9. Ken's Drawing.

Mrs. Clark: Ken, why did you use another diameter?

Ken: To see if they are the same.

Mrs. Clark: Is that important? What if he had it too far down like this. And Ken thought it was here. Is that going to make a difference? (Mrs. Clark draws (see Figure 10) a line lower in the circle that looks like a chord of the circle.)

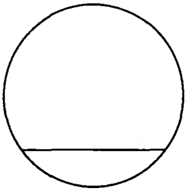


Figure 10. Mrs. Clark's Drawing of a Chord

Students: Yes.

Mrs. Clark: What happens to the circle as you get away from the center?

Denny: It curves and goes--it curves like this and goes down. (He uses his hands to show a smaller distance.) So that way the sides of the circle get smaller.

Mrs. Clark: The distance between points as we get away from the diameter gets smaller. Would you say that the diameter is the widest part of the circle?

Students: Yes.

Mrs. Clark: So he checked two places to see if the diameter was the same. Then you did what, Ken?

Ken: I halved 9 which is 4 1/2 and a marked it in the middle.

Mrs. Clark: What is the distance from the point which he is thinking might be the center to the edge of the circle called, Emmy?

Emmy: The radius.

This is not the first time they have heard about the parts of a circle. During the calendar section, Mrs. Clark uses the correct terms when they discuss the geometric cutouts.

Ken then measures his second diameter to see if they are the same length. He finds out they are the same. Mrs. Clark asks him to draw another diameter and measure it. When Ken does this, he finds that the measure is not 4 1/2. Mrs. Clark asks for another student to help out. Hal comes up erases the diameter and starts again. He decides that the radius is 4 and draws a radius and a point which he thinks is the center. When he does this, he decides that because one side is wider, 4 must not be exactly correct. He erases the point in the center and moves the ruler a little.

Mrs. Clark: "Why did you erase the half?"

Hal: It was to up here too high.

Mrs. Clark: How could you tell it was too high?

Hal: I looked at it.

Mrs. Clark: So why are you erasing it now.

Hal: It still doesn't look right.

Hal measures again and finds that the width of the circle is "9 inches and a mark." He draws several diameters and measures them to be sure they are the same length.

Mrs. Clark: And what are they supposed to do in the center?

Hal: All of them are supposed to go through the middle and cross.

Mrs. Clark: What is it when they cross at that point? What do we call that?

Hal: Intersect.

Mrs. Clark: How many diameters do you think that you have to draw, Hal? Till you decide that is the center?

Hal: (counts) 6.

Mrs. Clark: So you think that in order to find the center, we need to draw exactly 6 diameters? You couldn't do it with 5 or 4?

Hal: No, cause it wouldn't be enough.

Mrs. Clark then asks the whole class how many would be the least number of diameters that he could draw to check the center. After several arguments, most students decide that there should be at least two. Another student, Jim, uses a protractor to make his circle (see Figure 11). (The class on a previous day had worked with protractors when they were using their geoboards.) He outlines the protractor once, then flips it over and outlines it again.

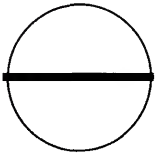


Figure 11. Jim's Drawing.

Jim says that this is a circle because it is 360 degrees. (They had previously solved a problem that showed them that a protractor was 180 degrees.) His diameter is the bottom of the protractor. He puts the center point through the whole in the protractor.

Emmy shows how she found the center of her circle by cutting it out and folding in half and then into fourths. The discussion continues,

Mrs. Clark: Now, how does what Emmy have in her hand different from what is up here?

Students: She didn't draw it.

Mrs. Clark: So it makes a difference whether or not we can pick our circle up and bend it. Which one is easier?

Students: Something we can bend.

Ann reads from her math log. She explains the steps that she has written out about how to find the center of the circle. The words that she has written explain essentially the same as the method that Gerry, Hal, and Ken have used.

Mrs. Clark has involved students in a problem that has helped them *understand* how to draw plane figures, in this case, a circle. They are learning the relationship involving the lines such as diameters and radii of a circle. In this context, the children have been learning about these relationships while being involved in solving what for them has become a real problem. They are learning about exactly what the center of a circle means. Obviously, from their solutions to this problem, they know that diameters must be congruent, and the radius is half of the diameter. They are attaching new learning to previous learning. They have related their learning to past situations when they explored diameters and radii of circles with their geoboards. They have learned through this experience that a radius is half a diameter. They have used this old knowledge to help them find the center of a circle. With this example, Mrs. Clark has made the students think about one of the important mathematical truths by asking, "How many diameters do you need to be sure that you have the center of a circle?"

Problem of the Day

The last part of mathematics class is the problem of the day. Most of the time discussion about the problem of the day will continue until the end of mathematics class. However, sometimes Mrs. Clark uses the problem of the day to introduce a new concept that they will explore after they have finished the discussion about each day's particular problem. Mrs. Clark explains her purpose for the problem of the day.

> [This] is the most thought provoking section. [It] is the section in which I really try to challenge them to think or reason. This was very difficult for most of them when we started the year. If it did not have an easy to arrive at answer, they simply would not attempt it. My number of 'takers' has increased steadily. Most now think of it as fun and see that 'for real' I am not as much interested in the answer, but rather how they arrived at the solution they got.

The students have worked independently on the problem since they got to school at 7:40 or 7:45. Now, they are discussing it as the last part of their mathematics time. The problem is as follows:

> Is it possible to make a solid figure on a geoboard? If yes, explain how. If no, explain why not.

Mrs. Clark: Pat, what do you think?

Pat: No. Because if you put rubber bands on there it is not going to be a block.

Mrs. Clark: What does being like a block have to do with being a solid figure?

Pat: Well, if you want a solid figure, you have to make a block. Or then it is just plane. It is just a plane figure.

Mrs. Clark: Anybody else? Jim? (Many students are saying, "Oh. Oh. Oh." with their hands up.)

Mrs. Clark prods and probes to get students to contribute their ideas and clarify their thinking. She says, "Tell me more." "What do you think?" Other students answer,

Jim: No, if it is a solid figure, there's not going to be air in it. You can't do that on the geoboard.

Denny: No, because you can't make the sides go up here on the geoboard and hold it in your hand.

Eddie: On a geoboard, it is flat, and it's not three-dimensional.

Mrs. Clark summarizes the students' reasons and decides to change the way she has worded the problem. By rewording the problem, she stimulates students' thinking.

Mrs. Clark: So you are saying, "No, because in order for it to be a solid figure it cannot be flat. It has to be three-dimensional?" Now, I think I have made a mistake. I want to change one of my words up there. I want to say, "Is it possible to *represent* a solid figure on the geoboard?" Maybe that makes a difference in your thinking--maybe not. All right, Mary?

Mary: Yes, because if you just keep on stacking them around. It won't be flat anymore. You'll have more rubber bands and you can fill the whole thing in?

Mrs. Clark: When we talk in mathematics about a solid figure, does that mean it has to be a completely filled in space?

Students: Yes.

Dan: Because if it goes up and up and up, it's not flat. I stacked my rubber bands up from the bottom of the nail to the top to make it not flat.

Mrs. Clark: To make it what?

Students: Solid.

Eddie: Three-dimensional.

Mrs. Clark: It makes it three-dimensional. Danny's figure has length, width, and *height.* Look at Sharon's. She has got something other than a square shape represented. She has a triangular shape. Hal also has a triangular shape represented that has length, width, and height.

Mrs. Clark highlights Eddie's answer by saying what three-dimensional means. Then, she uses other students' representations on the geoboard to push the point. However, some of the students still disagree.

Ken: But there are still spaces.

Mrs. Clark: Thank you, Ken. You are exactly right. I agree with you with what you said. It is difficult to make it on the geoboard without spaces, isn't it? That's why I changed my word to *represent.* Ann is going to show you something. She has a different idea for representing a solid shape.

Ann holds up her geoboard. She has used rubber bands to make her figure look like this (see Figure 12).

Mrs. Clark: What shape does that represent?

Students: Cube.

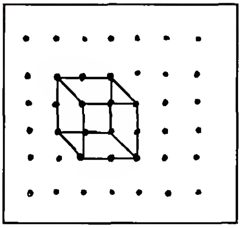


Figure 12. Ann's Geoboard Solid Figure.

Mrs. Clark: A cube. It does look three-dimensional like that, doesn't it? All right, raise your hand if you think now that after listening to these people's explanations that the answer is no. (Students' hands up.) Raise your hand if you think that he answer is yes. (Students' hands up.) If you raise your hand and say yes, I want you to show me on the geoboard. Make sure that what you are showing me *represents* a solid figure.

Although the context of this lesson is geometry, this problem is not just about definitions of solid shapes. The students have been exploring and investigating through problem solving. Mrs. Clark is leading them to think abstractly through visualization. The fun of trying to develop their thinking by using the geoboard was evident with the involvement of the students in this activity. Through hearing the students' thinking, she realized that she needed to change the word *make* to *represent.* She changed only one word to get them to think deeper. At the conclusion of the activity, Eddie, one of the students, decided that he would answer the question with a yes only if the word *represent* were used. He was adamant that if the word *make* was used, the answer would have to be no. He said that Ann's representation of the cube on the geoboard had convinced him. This activity demonstrates the possible interaction that can happen when students are allowed to verbalize their thinking without risk. Mrs. Clark later told me,

> After you left, we got into a discussion. Mary said, "Mrs. Clark, you never did circle the answer up there, yes or no." I almost always circle the answer. So I said, "You are right, I didn't. I wonder if there is a reason that I did that?" We discussed again why some people said it would be yes and some said it would be no.

Mrs. Clark thinks that the students in her class have shown remarkable progress in understanding mathematics. She credits “the problem of the day” for much of this progress. Students tell her how much they enjoy mathematics in her class. They become involved in the lessons and talk about what they have learned to others. Mrs. Clark said, “Erin said she told her mom that such and such was a solid object.” Her mom said, “No, it is hollow.” But Erin said, “No, you don’t understand. In geometry, it is a solid. It’s three-dimensional.” There is more communication in the classroom than one sees. Students are not only communicating in the classroom, but they are also taking ideas home with them and sharing them with their parents.

Mrs. Clark says the problem of the day is the “one which most involves the parent. I encourage the students to take home some of the problem-solving situations they have enjoyed most and try them on their parents.” When parents have come to conferences, they often want to know about things that they hear their children talking about. Mrs. Clark says, “They’ve told me they want to see this ‘calendar and money thing we do every day.’ They share their children’s comments about some of the problem-solving situations.”

Summary

With this chapter, I have provided a typical day in Mrs. Clark’s mathematics class. From the excerpts of the classroom interaction, we can see that Mrs. Clark provides students with opportunities to speculate, examine, and justify their solutions to problems. In an effort to help them gain the power to think mathematically, she gives them experiences that help them learn to use higher-level thinking and ideas. Her students have learned to create mathematics at this higher level rather than learning lower-level rote and computation skills. She provides them with opportunities to understand that mathematics is about making sense of things.

As I observed this mathematics class, I was amazed and excited by the students’ theories and predictions. I was pleased and encouraged by the higher-level thinking and the processes by which they arrived at solutions. As I watched their progress through the school year, they exceeded my previous expectations of fourth-grade mathematical thinking.

CHAPTER V
KEY ELEMENTS OF INSTRUCTION WITHIN MRS. CLARK'S TEACHING

Teachers determine how they will teach mathematical content to students. They decide upon the strategies they will use to engage students. They create the opportunities for the students to learn the knowledge and skills that will help them gain mathematical power. Mathematical power as defined by *The Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989) "is the individual's ability to reason logically, make conjectures, and communicate effectively about mathematics" (p. 5).

Mrs. Clark believes that instruction should be guided by the students' learning. She organizes her teaching in such a way that she is always answering her own question, "What do I want my students to take away from this experience?" She wants her students to think mathematically and see the whole picture of how mathematical concepts are related. She chooses experiences that achieve these goals. Her teaching primarily involves students in investigating relationships among concepts, in discovering relationships of manipulative materials to concepts and procedures, and in participating in classroom interaction. Mrs. Clark continually says, "Tell me what you are thinking." She helps students build on their informal knowledge and guides them to develop a meaningful relational understanding of mathematics that connects the concepts and the procedures. Mrs. Clark provides learning experiences in such a way that students can actively construct correct mathematical meanings and mathematical power. As she teaches, she looks for signals to inform her when the concepts she is teaching cause difficulty. She constantly evaluates whether the lesson is proceeding in a positive direction for her students' learning. She then adjusts her teaching to the students' needs.

In this chapter, I will describe the 10 instructional strategies that Mrs. Clark uses to help students develop mathematical power and relational understanding of mathematics. These strategies are consistent with Piaget's research and theory called constructivism. Piaget emphasized that children acquire knowledge by constructing it from the inside and not by internalization from the environment (Kamii, 1989). Mathematical knowledge that is rich in relationships must be created by the individual (Piaget, 1973). I include excerpts from classroom discourse and Mrs. Clark's interviews to explain how she uses each strategy. I end the chapter with a summary of how these strategies help students understand that mathematics is "a way of thinking."

Mrs. Clark's Instructional Strategies

I am going to talk about separate strategies as a way of analyzing Mrs. Clark's teaching, but, in fact, these strategies are not separated from one another. As you read, you will see that I have developed some strategies that appear to be more about communicating or about building knowledge or about assessment. I use these classifications only to help us think about the strategies. They are not separate entities but are intermingled. Even though I have sequenced some strategies or assembled them together in some order, they are not hierarchical. Mrs. Clark simultaneously uses these strategies as she teaches.

Mrs. Clark's teaching is a balance between attention to conceptual understanding and application. The students engage in the problem situations by discovering relationships and patterns while they search for answers. The teacher guides the students to discover mathematical rules for themselves by asking them to present and justify the solutions that they create for problems or algorithms. She sometimes provides clues that guide the students toward concepts, procedures, or problem-solving abilities. She asks the students to apply their learning to new situations. She demonstrates that learning to think can be engaging, enjoyable, and empowering.

Uses Concrete Problem-Solving Contexts

The key to Mrs. Clark's mathematics teaching can be found in the problem-solving experiences that she provides for the students. Mathematical problem-solving is the center of all the learning of the content in her class. The problem-solving experiences include a variety of word problems, strategies for solving problems, problems using manipulatives to model thinking, open-ended problems, and everyday real-world problems. In this way, Mrs. Clark helps students to see that mathematics is everywhere. Mrs. Clark does not teach skills to be learned *for* problem solving but teaches skills *through* problem solving. In her teaching, problem solving is not a distinct topic separate from other mathematics learning. Hiebert and Carpenter (1992) wrote,

> The implication is that learning situations should be embedded in authentic problem situations that have a meaning for the students. By learning concepts and procedures in problem-solving contexts, it is presumed that the knowledge is connected so that it is accessible for problem solving. (p. 81)

Ms. Clark organizes most classes around problem-solving experiences that allow students to find solutions to nonroutine problems. In this way, she involves students in *doing* mathematics. Kantowski (1981) defined nonroutine problems as "problems for which a problem solver knows no clear path to the solution and has no algorithm which can be directly applied to guarantee solution[s]" (p. 114). Mrs. Clark involves students actively in the learning process. Von Glasersfeld (1981) wrote that active learning is taking new information and acting upon it in some way to create meaning. Students construct knowledge if they are actively involved in solving problems that they understand and *want* to solve. In this class, students do not sit passively and watch the teacher perform actions to solve the problems but solve the problems themselves. According to the constructivist view of learning, based on Piaget's theory of assimilation and accommodation, students must *experience* the concepts in order to build understanding (Davis, Maher, & Noddings, 1990). In Mrs. Clark class, students become involved in experiencing mathematics. Even though Mrs. Clark's mathematics class usually lasts from an hour to an hour and a half, several times during my observations students became so involved that they asked to skip recess so that they could continue with their mathematics.

Activities grow out of everyday situations. Mrs. Clark asks open-ended problems that have several answers. Students explore relationships or create solutions with their own methods in open-ended situations where there are many ways to find answers. Their experiences of learning the content help them understand that mathematics is a process of learning to think, not a product.

Mrs. Clark presents problem-solving situations that arouse students' interest and become problematic for them. Mrs. Clark's problems cause them to examine their current knowledge. Kantowski (1980) noted that a problem is a situation for which the individual has no available algorithm to use but must put together their knowledge in a new way to solve the problem. According to Yackel, Cobb, Wood, Wheatley, and Merkel (1990), teachers should provide children with activities that are likely to give rise to genuine mathematical problems. Piaget (1973) believed that the teacher should stimulate the child's mind with appropriate tasks.

Mrs. Clark tries to insure that students develop the thinking processes that will give them the foundation for solving problems. She guides the students to develop logical thinking through looking at patterns and relationships. She helps the students see how the numbers in mathematics are developed through their relationships with each other. They learn problem-solving strategies by making tables, charts, and graphs; by making lists; and by drawing pictures and diagrams. They learn how to guess and check and make models of their thinking. These experiences help students construct problem-solving tools for thinking about ways to find solutions.

Mrs. Clark never hurries through problems but concentrates each day on a few significant problems. The problems demand conceptual knowledge and higher-order thinking strategies from her students. She emphasizes primary concepts that are central principles that define mathematics; for example, addition and subtraction are inverse operations; a fraction is a part of a whole; a fraction's value depends on the whole to which it refers; and a circle has only one center. According to Brooks and Brooks (1993), problems that are structured around primary concepts or conceptual clusters provide a context in which students learn to gather and analyze information that relates to the whole concept rather than to learn the steps in a linear fashion.

Mrs. Clark's problems have a clear purpose. Her purpose for each day makes each lesson or experience focused and coherent. Mrs. Clark does not sequence her instruction from part-to-whole. She does not break the problems down into steps for the students. She presents the problems or activities in a conceptual context where students learn to look at the whole conceptual relationship or meaning of the problem. Mrs. Clark guides the students to look at the whole in order to break it into parts. She does this so that students understand the parts they have created and how the parts relate to the whole concept.

Mrs. Clark helps students understand the problems by finding out what information is given and what is being asked. She helps them construct plans that could lead them in new directions toward thinking about getting the answers. Brooks and Brooks (1993) suggested that when students construct the process required to solve problems, rather than having it done for them, they learn to make sense of information.

At least half of the daily time for mathematics is for whole-group problem-solving activities. Geometry is the primary content for these problems. Mrs. Clark discusses why she used geometry as a basis for much of her class discussion.

> Geometry is all around them. There are lots of real-life things that involve geometry. They need experiences with geometry, like with tangrams, so that they are improving their thinking. They might not technically need them now, but this thinking will be beneficial to build on later. It will give them some spatial concept kinds of experiences. So that later on, they don't have to go through that step. They've already learned how to think and can see things more clearly. They need . . . these experiences as building blocks for future things that they will do in math. [These] experiences will also help them right now with things that they might want to know how to do.

Mrs. Clark says that her students knew little about geometry when they came into her class. Therefore, she can present problems conceptually to them without having to overcome misconceptions that they had earlier formed about the content from working and memorizing procedures. Mrs. Clark says that building their mathematical thinking is perhaps less difficult in geometry than it is in some other topics such as division and fractions. Hiebert and Wearne (1986) have supported a similar idea. These researchers concluded that if children are taught procedural knowledge before they understand the concepts, it is very difficult for them to link the two together.

Good examples of concrete problem-solving activities are two of the problems that I discussed in Chapter VI, finding the center of a circle and exploring whether a solid figure can be represented on a geoboard. These problems were open-ended with no set method that the students had to use for solutions. The students became genuinely challenged while they were *doing* mathematics. They actively explored the problems and discussed their ideas. Finding answers to these problems involved students in learning about primary concepts of mathematics, the concept of the center a circle and the concept of solid figures. In both problems, Mrs. Clark asked students to think conceptually about the problem and guided them in looking at the *whole* in order to see the *parts*. They explored the idea of a whole circle. By trying to find the center of the circle, they had the opportunity to explore the parts of a circle--the diameter, the radius, the circumference. From discovering the parts, they have learned to understand the quality of the whole. When students explored how to represent a solid figure or if they could represent a solid figure on a geoboard, they looked at the whole concept. By trying to represent the solid figure, they had the opportunity to learn more about the three dimensions of length, width, and depth.

Assesses Students Continually

Assessment is a significant component of Mrs. Clark's instruction. Assessment of her students' learning is interwoven with her teaching. Mrs. Clark's assessment is authentic. This means that she assesses students in meaningful activities that are relevant to them. She observes the students' actions. According to Carpenter and Fennema (1991), teachers learn more about what students know by observing students' behavior. Fennema and Carpenter (1992) stated that the primary reason for assessment is "to decide whether a task has meaning for a child" (p. 45). Mrs. Clark believes that she needs to know if each child understands what he or she is studying. She talks about how she assesses their understanding. She says, "I listen closely to what they say. Each child is different. Even though some of them are not as clear as others, I think it is important that I know where every child is coming from."

Mrs. Clark uses a variety of assessment methods. She listens to the students' explanations. She interprets students' responses. She evaluates group work. She evaluates individual work. She observes students' behaviors. A method that she often uses to evaluate the students' conceptual knowledge is to provide a problem or task that they have never seen before. However, the context or situation of the problem is one that is familiar to them. For example, as an extension of the lesson when the students found the center of a circle by using several different ways, the next day Mrs. Clark's problem of the day was as follows: What is the perimeter of the square (see Figure 13)?

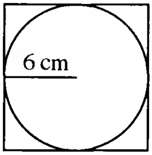


Figure 13. Circle Inscribed in a Square

Mrs. Clark is assessing with this problem whether the students have connected the work they had been doing to finding the center of the circle. She wants to know if they have connected their knowledge of 6 centimeters being half of the diameter of 12 centimeters with adding the 12 four times to get the perimeter.

Assessment is an ongoing process in Mrs. Clark's classroom. She assesses student learning to discover the information she needs to make her instructional decisions.

It is linked continually to her instruction. Chambers (1993) noted that "every instructional activity is an assessment opportunity for the teacher as well as a learning opportunity for the student" (p. 17). Most of her assessment is informal such as listening to students' responses during the oral discussions or during cooperative learning activities. According to Fennema, Franke, Carpenter, and Carey (1993), assessment should be a regular part of instruction. Children should regularly be asked to explain how they figured out answers. Since most of Mrs. Clark's mathematics instruction is done through questioning and answering during whole-group discussions, interpreting students' responses and observing their behaviors are her most frequent methods for evaluating her students' learning. She talks about her assessment during classroom interaction in the following excerpt:

> I take into account and consideration [the students' contributions] during the interaction in class. That's important because that's part of what we are doing. That has to be evaluated. It is more subjective. I don't always make an immediate evaluation on a sheet of paper right after the math lesson about who did what. I store it in my head.

Mrs. Clark knows the right questions to ask to evaluate the students' thinking and reasoning. Fennema and Carpenter (1992) believed that teachers must have knowledge of problem difficulty and distinctions between problems so that they can adjust instruction. By asking the types of questions that require students to think, Mrs. Clark learns what her students know. She often says, "What would happen if . . . ? Can you predict . . . ?" During whole-group discussions and individual questioning of students, she asks them to model their thinking with manipulatives, diagrams, pictures, or by whatever means they can. Because she is looking for their understanding of concepts, not only procedures, this assessment takes time. During class she walks around the room looking at students' logs and asks them how they solved the problems. She collects the students logs and evaluates their work. She talks about her informal assessment in this way.

> I circulate while they are working [in their math logs] and make observations about how they are progressing. And I am very attentive to who is participating during the discussion of each section. Based on these informal observations, I mark an E (excellent), S (satisfactory), or N (needs improvement). I also collect the math logs about every two to three weeks to evaluate. It is not graded according

to right or wrong answers but rather it is evaluated based on completeness and amount of effort that is reflected.

Mrs. Clark evaluates in ways that are compatible with her teaching. Because she emphasizes conceptual knowledge when she teaches, she uses assessment methods that help her understand her students' knowledge of concepts. She teaches mathematics to her students by asking them challenging problems; therefore, she also grades the students, both cooperatively and individually, on their solutions to challenging problems. Because Mrs. Clark's teaching is rich with connections, relevant to students, and authentic to real-life, the questions on the oral or written assessments do not have predictable answers but require higher-order thinking and often the use of manipulative materials.

Mrs. Clark focuses her assessments on students' learning as they do mathematics rather than simply checking for right answers. Her methods of assessment signal to her students the parts of mathematical learning she considers important. Through her assessments she communicates to students the idea that mathematics is not about getting answers but about learning to think. Her assessments of her students are rich rather than superficial. They lead students to think about new problems. These assessments stress problem solving and the use of manipulative materials. Lower-level mathematical skills should not be the focus of assessment (Lambdin, 1993).

Mrs. Clark plans her assessments to evaluate what the students know. She says she thinks that the kinds of activities she asks her students to do are in line with where they are thinking. She says, "You try to assess as close as you can to get to the child's understanding." To get close to their understanding, she often asks them to model their thinking with manipulative materials. Manipulative materials are "of major use in helping teachers understand what the child is thinking" (Fennema & Carpenter, 1992, p. 69). Cognitively Guided Instruction recommends that children use manipulative materials to help them show how they think so that teachers can plan their instruction accordingly. Mrs. Clark constantly monitors student understanding and adapts lessons to this understanding.

Assessing cooperative groups. While students work in cooperative groups, Mrs. Clark walks around and observes if all students are working and discussing whatever task she had given them to solve. As she moves from group to group, she listens for the reasoning of the students and for difficulties they may be experiencing. Newman, Griffin, and Cole (1989) discussed this factor.

> Instead of giving [students] a task and measuring how well they do or how badly they fail, one can give [students] the task and observe how much and what kind of help they need in order to complete the task successfully. In this approach the child is not assessed alone. (pp. 77-78)

Mrs. Clark listens to student dialogue for another reason. She wants to be sure that all students contribute to the thinking and discussion of the group. Mrs. Clark says, "As I walk around, I look to see if a particular person is not doing as much of the thinking. Then I can question within the group and try to bring his or her thinking out." One way that she makes sure that everyone participates in group work is by making sure that there are enough manipulative materials for everyone. Some examples are the following measurement activities. Mrs. Clark made sure that there were enough objects for each person to measure the length or mass or whatever the problem was asking. She also says, "I try to make sure they are taking turns."

During cooperative learning activities, Mrs. Clark assesses the students both within their groups and individually. For example, as a cooperative-group assessment on a linear measurement activity, she asked students to work together solving some problems that she prepared on a worksheet. These problems are situations that students will encounter in real-life.

1-Mrs. Clark wants to frame the St. Augustine print. The frame she has chosen is 2 inches wide. It cost 15¢ an inch.
  a) How many inches of framing does she need?
  b) How much will it cost?

2-Terry is going to put a decorative trim on 4 towels. The trim costs $.89 a yard.
  a) How much trim is needed?
  b) How much will it cost?

3-Jordan is building a dog pen with the same dimensions as your cardboard box. But 1 inch = 1 foot. The fencing he is using costs $5.50 a foot.

a) What are the length and width of the dog pen?
b) What is the perimeter of the pen?
c) How much will the fencing cost?

In an interview, Mrs. Clark talked about how she assesses a cooperative-group product.

> I try to set it up so that the group turns in one product with their names on it. Everyone will get the same grade. I try to monitor what is going on in each group. I want to see if there's a problem in any group and [whether] it looks like they are headed in a positive direction. If they are not, that's going to hurt their grade. If they are all working in the same direction, turn in a poor product, and they seem to be content with it, they all get the poor grade.

She is not only assessing their thinking and work, but she is also emphasizing that the students must cooperate in order to get a good grade. When the students do cooperate, but prepare a poor group product, they all get the same low grade. Occasionally, a student in the group will not agree to turn in a poor product, but he or she cannot get the group to care about the work. Then, Mrs. Clark allows the student to turn in individual work. She explains this in the following interview excerpt:

> Like Patty Ann, in particular. She is really precise. She was not satisfied with her group's product. . . . So, I told her, "You must cooperate with your group and give input, but if you want to do something outside of your group and turn that in as your product, that is fine." She wanted a higher quality product than her group was aiming for.

Sometimes a student might really disagree with the group's answer on a problem but cannot get the others to change their minds, no matter how good the argument. When this happens, Mrs. Clark provides an out for the student:

> If a student will try to convince his/her group that an answer is right, and they don't want to accept that argument, then he/she may want to go ahead and put down a different answer. Then I allow [the student] to turn in a separate sheet.

Individual written assessments. Individual assessments are similar to the types of activities that students solve within their groups. After students complete problem-solving worksheets in their groups, Mrs. Clark brings them back to the whole group, and they discuss the various strategies for finding solutions. Then, Mrs. Clark gives the students individual assessments in which they must use some of the strategies they have learned in their group work and on group assignments. On these assessments, Mrs. Clark orally asks the questions while the students work the problems on large 11 X 14 manila newsprint blocked off in 10 sections for 10 questions. She asks them to show their thinking by drawing pictures and diagrams. The problems are just different enough from group assignments that the students can not memorize the steps or algorithms to solve the problems. Mrs. Clark repeats the questions many times and gives the students a long time to think about each problem before she goes on to a new one. One example of an individual written assignment is a linear measurement test. Four of the questions are very similar to the cooperative group problem solving worksheet. Other questions assess their estimating ability--an area that she has emphasized during all the activities. The following are some of the estimating questions that she asked the students.

1-Which leash is 6 feet long--the purple or blue (holds up two leashes)?

2-Which picture frame is 8 inches by 10 inches (holds up two frames)?

3-There are 3 speakers in this (holds up a stereo speaker component). If one needs to be replaced, I can't go up and say, "I need a speaker." I have got to tell them something about what size it is. Which speaker has a 4 inch diameter, top or bottom?

Obviously, the types of questions that Mrs. Clark asks help her understand not only whether the students can measure but also whether they have developed, through estimating and measuring, a number sense for distance. The NCTM (1989) emphasized both estimating and measuring in helping students develop a measurement sense.

Mrs. Clark's qualitative methods of assessment take time. However, she is able to save time by not having as many written assignments. In order to have that time, she says, "You have to reduce the amount of written work." Mrs. Clark says that too many teachers emphasize written work and think "they have to have a written work grade every day, a daily grade, for math." Nevertheless, Mrs. Clark does not just test students at the end of chapters or units. About once a week, she sets aside a separate time for a paper-and-pencil assessment. For example, she might ask students to compute five division problems or ask them to work five problems with adding fractions. These quizzes are always very short. About once a month, students take a long test somewhat like the chapter tests in the textbook. With this chapter test, Mrs. Clark also provides experiences for students to learn to take standardized tests.

Encourages Students to Communicate

Seeking to understand what students are thinking and how they think is central to constructivist teaching and learning. Brooks and Brooks (1993) wrote,

> The constructivist approach to teaching presents . . . real-world possibilities to students, then helps the students generate the abstractions that bind these . . . together. When teachers . . . ask students to describe [their thinking] . . . they encourage students to analyze, synthesize, and evaluate. Learning [is] the result. (p. 104)

Mrs. Clark and her students work together to build meaningful discourse in which students learn to think mathematically. Several researchers have identified discourse as a learning activity that facilitates constructing knowledge and improving thinking (Cobb, 1994; Cobb, Yackel, & Wood, 1991, 1992; Corwin & Storeygard, 1992; Lampert, 1990, 1991; Vygotsky, 1978). Cobb, Wood, and Yackel (1990) called the process of communicating and negotiating to create meaning *social constructivism.* Listening to students' reasoning and ideas about solutions is the clearest way of understanding their thinking. Mrs. Clark says, "I think [the students] learn to understand by talking about their thinking and trying to explain it. That's why I would like them to explain it to another person. Understanding another child's answer helps move them to a higher level in their own thinking." Through social interaction, students learn to interpret others' perspectives and fit them with their own interpretations. Kamii (1989) noted that "other students' ideas are important because they provide occasions for children to think critically about their own ideas" (p. 32).

In this fourth-grade classroom, Mrs. Clark asks students about their thinking and solution strategies rather than just accepting their answers. She asks them to explain and justify their answers. Mrs. Clark says, "I want them to come to an answer that they can justify because if they can justify it, it usually has some validity as an answer. When children are taught to think things through and to justify and explain, it becomes a habit."

She helps students develop habits of thinking, analyzing, and verbalizing their reasoning as well as understanding and analyzing the reasoning of others. According to Lampert (1990), students should make conjectures and communicate solutions, try to convince themselves and one another of the validity of particular solutions or answers, and rely on mathematical proof and argument to determine the validity of answers.

Mrs. Clark requires her students to communicate and explain their thinking. Over time the students learn to explain their thinking so that other students can understand their ideas. They become excited when other students understand their explanations. Mrs. Clark says,

> Communication is important. I try to give them the opportunity. It makes them feel good when they can say what they understand so that others are convinced. They feel challenged when they have to explain their thinking. Sharing clarifies their thinking. It lets them verbalize . . . instead of just having it in their mind.

Mrs. Clark rarely just tells students answers because she wants them to clarify their own thinking and understand other students' thinking. Kamii (1989) suggested that the role of the teacher is to help the students clarify their statements. Mrs. Clark does not assume that when the students give correct answers they have constructed correct knowledge. She helps them learn to agree and disagree, and prove their answers. Finally, Mrs. Clark leads them to decide if their answers seem reasonable. She helps them learn to distinguish between invalid and valid arguments. According to Lampert (1990), the content of mathematics lessons should be in the arguments and dialogue in the classroom.

Mrs. Clark helps her students to become aware of *how they think* and to verbalize their thinking so that she can better help them with any difficulties that they are experiencing. Whether the help comes from her or other students, she wants them to get the help that they need. Mrs. Clark says,

> I can't help them with a wrong answer if I don't know where they are coming from. I am getting them to practice verbalizing or explaining so that when there is a problem that has a direct answer, they are better at getting me or their next teacher or whoever to understand what it is that they can't solve.

It is the responsibility of the teacher to try to figure out what the student means and to assist him or her in verbalizing this meaning (Yackel et al., 1990). The most frequent source of confronting and considering one's own knowledge is through interacting with others (von Glasersfeld, 1989).

Mrs. Clark guides the discourse so that students feel safe expressing their thinking. She creates an environment in which they verbalize their ideas and explanations without risk. Mrs. Clark teaches them to negotiate and collaborate through accepting their responses to questions or problems without judgments of right and wrong. Bruner (1986) said that learning to negotiate is learning to construct new meaning by which individuals can relate to each other. During the classroom discussions, Mrs. Clark's students show a willingness to question each other's answers. She says, "I am listening to *how* they listen to each other." Mrs. Clark provides students the opportunities to learn that there can be more than one solution to problems and more than one way to arrive at those solutions. According to Yackel et al. (1990), children at differing conceptual levels use different solution methods and interpret tasks in different ways. Mrs. Clark says, "We all hear things in a different way. If students don't understand it with one student's words, maybe they will with another's. It is all good information." Students discuss their solutions with other students, not just with the teachers. Students listen and question each other until they understand each other's solution strategies. The process of having students explain their solutions is repeated again and again. Cobb et al. (1990) found that "opportunities for students to construct mathematical knowledge arise as they interact with both the teacher and their peers" (p. 137).

Mrs. Clark emphasizes methods over answers, and she leads the students to concentrate on methods rather than answers. Mrs. Clark often asks, "How many ways can you solve this problem?" or "Did anyone solve it a different way?" She believes that understanding comes from crossing the same territory in many different directions. Mrs.

Clark discusses students' different solution strategies in the following excerpt from an interview.

> Even though they may not understand one student's approach, they understand how they could get it in a different way. I stress this. Ken's way of thinking about something might be different than another child's way. Like with Erin, she was adding 12s over and over. I wanted her to go ahead and add the 12s until it got inconvenient for her to add. Then she would multiply. If they don't do the problem like Eddie does it or like April or like Patty Ann does it . . . that's okay.

The students learn to use their minds and become very articulate at expressing their thinking. The following example in which Richard tries to articulate his thinking about his approach to a fraction problem demonstrates how Mrs. Clark gives students the opportunity to learn to articulate and clarify their thinking.

The day after students had been working in groups of five using fraction circles to subtract fractions with like and unlike denominators, Mrs. Clark wrote five problems on the chalkboard. One of them was (2 - 1 1/2) + (3/4 - 1/4). Many students had their hands up. First, she asked students to give only their answers. Students gave several answers--124, 12, 5/4, 1 whole, and 4/4. Afterwards, she went back to get their explanations. She asked one student, Richard, to explain his answer of 4/4.

Richard: 3/4 take away 1/4 = 2/4. Then, 2/4 and 2/4 = 4/4.

Mrs. Clark: I understand the first part that you did. 3/4 - 1/4 = 2/4. Can you explain how you got the other 2/4?

Richard: If you broke it into four parts and took half away.

Mrs. Clark: What did you break into four parts?

Richard: 2.

Mrs. Clark: Then you took out half?

Richard: Yes.

Mrs. Clark: You said that you added 2/4 + 2/4. Why did you add 2/4 to the 2/4 that you had over here (pointing to the 3/4 - 1/4)?

Richard: Because it said to add 1/2.

Mrs. Clark:	I know where you got it from, Richard. But I want you to be able to explain it. Where did you get this 2/4 from (pointing to his second 2/4)?
Richard:	The half--because 2 - 1 1/2 is 1/2.
Mrs. Clark:	Okay. Then how did the 1/2 become the 2/4?
Richard:	If you say 2/4 is a half, then 2/4 and 2/4 = 4/4.

Mrs. Clark supported Richard in clarifying his thinking. Mrs. Clark was convinced that Richard knew what he was doing, but she persisted in questioning him until he provided a clear explanation. She wanted him to learn to articulate what he was thinking. If she could get him to say what he was thinking about solving the problem, then hopefully, he would reinforce his own thinking and help others understand the problem more clearly. Mrs. Clark later said, "They have to say it. They have to say words to explain [how they think]."

By having Richard explain his thinking, Mrs. Clark adroitly directed the students to compare their method with his and implied that his method is important for them to understand. By bringing her attention to Richard's idea, she asked the students to do the same. Through her questions, Mrs. Clark asked Richard (and the other students) to identify their choices and articulate their reasons for making them. She says, "They have come a long way in being able not just to accept any answer but to think through the answer. Whether or not an answer makes sense is an important place to be."

In an interview, Mrs. Clark defined her type of teaching. She said, "Interaction--a balance between the students and me." The ongoing discussions in her classroom show that, indeed, her definition of her own teaching as interaction plays out in the balance of communication. Students spend a great deal of time talking to one another about mathematics. However, the students communicate their thinking not only to the teacher and to each other but also to their parents. Parents often come in to tell Mrs. Clark things that their children have said about the mathematics they are learning.

Values Students' Thinking

Mrs. Clark creates a learning environment that allows students to gain the confidence to construct knowledge that is powerful and correct. Mrs. Clark affirms her students' thinking by remembering what they say and using their dialogue to build the lessons. She shows that she values the students' approaches to problems by recognizing them and speaking about them. She calls students by name to get them involved and keep them involved. She listens closely and repeats their answers to the other students. As she listens closely, she provides a model for her students to follow.

Mrs. Clark makes each child feel that his or her answer has worth by allowing each one to verbalize solutions or strategies. Mrs. Clark encourages the students to take risks. In this way, students build confidence in their thinking. Mrs. Clark says,

> Sometimes I paraphrase it (an answer) because I want the other students in the class to understand it. But I also want the child to know that I am understanding and really listening to what he or she says. If they don't agree that I've done it correctly, they will respond back. I want the other children to hear the idea and value each other's thinking.

By listening carefully to her students' ideas and having other students do the same, Mrs. Clark validates the power of their thinking. Cobb et al. (1990) noted that when students express their thoughts, the teacher should be expected to respect their thinking and require others to also do so.

Mrs. Clark is committed to engaging *every* student to contribute toward the thinking of the class. During her teaching, Mrs. Clark does not exclude those who have solved problems incorrectly. Every day, at some point during each lesson, she asks all students to explain their thinking.

Notice in the following lesson, as with the others I have presented, that Mrs. Clark often repeats the students' answers. Sometimes she repeats them word-for-word. Other times she paraphrases their answers. Note how she gives credit to students' ideas as she uses students' contributions to connect their ideas. Mrs. Clark begins the lesson.

Mrs. Clark: Can more than one circle have the same center?

Mrs. Clark: Raise your hand if you think that more than one circle can have the same center. It looks like we have about half and half.

Hal: Mrs. Clark, can I ask you one thing?

Mrs. Clark: Yes.

Hal: If it's like a circle that goes around and has a point in the middle--Can every one that goes around have the same middle of the circle? (He uses his hands to show one circle in the upwards direction and one circle sideways.)

Mrs. Clark: That's an excellent question and I didn't think that anybody was going to ask that. I think what he is basically asking--I wish I had something to show it with--I think what he is asking is if he has a circle like this. . . .

Mrs. Clark draws a figure (see Figure 14).

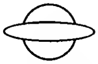


Figure 14. Drawing of Circles in Two Planes

Mrs. Clark allows students to ask questions of her. Mrs. Clark listens closely to Hal. By validating Hal's thinking, she helps him believe that his thinking has value. She values his thinking so highly that she helps him model and explain it to other students. Mrs. Clark gets two rubber bands and loops one around two fingers and another around two different fingers. She overlaps the rubber bands (see Figure 15).

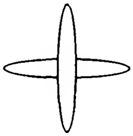


Figure 15. Overlapping Circles Using Rubber Bands

Right, Hal? And the center is in the middle of it. What he is wanting to know is what if I have a circle that goes like this. Is that right?

Hal: Yes.

Hal's idea concerns two circles that are not in the same plane. Some students disagree with his idea, but Mrs. Clark says she did not say that the circles had to be in the same plane.

Mrs. Clark: So Hal has thought of an example of two that do have the same center. One circle is headed in this direction in one plane and one in this direction in a different plane.

Hal: I came up with a good one.

Hal has become confident about his thinking. He knows that his thinking is valuable and powerful. According to McLeod (1991), students must believe in their own success in order to lead to greater understanding in their mathematical thinking.

Mrs. Clark: Can any body think of another way that is possible for more than one circle to have the same center. No one? I saw some diagrams in some people's math logs. Don't be afraid to share your answer. Ann, I saw yours. Come up here and show us your answer.

With this statement, Mrs. Clark shows she has been closely observing her students' work. She supports Ann in presenting her idea. Ann draws her representation on the chalkboard (see Figure 16). Mrs. Clark wants others to understand Ann's answer so she asks several students to come up. One student becomes the center and others hold hands to form

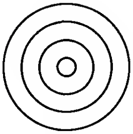


Figure 16. Ann's Drawing

several different circles around the center student. After this demonstration, Missy has a new idea and shows it on the chalkboard (see Figure 17).

Mrs. Clark: These are called concentric circles. Concentric circles.

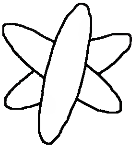


Figure 17. Missy's Drawing

Mrs. Clark remembers Hal's idea and shows how it is connected to Missy's idea. She recycles Missy's answer to help students remember a solution that they have already heard.

Mrs. Clark: Missy is kind of expanding Harold's idea. She is saying they don't have to be in perpendicular planes, like Hal's. She is saying that I could turn this direction at a slant, move over a little and make another one and so on. Right? Is that still okay?

Mrs. Clark gets out a sphere and puts several rubber bands around it (see Figure 18).

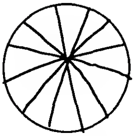


Figure 18. Connecting Missy and Hal's Ideas

Mrs. Clark: I could have the circle here. (She puts another rubber band.) One here. One here . . . How about that. Isn't that neat? All the ways that I could turn the circle would all have that same . . .

Students: Center.

Gerry: There's umpteen times that you could turn that.

Mrs. Clark: What is another word we could use besides umpteen?

Some students: Infinite.

Mrs. Clark: Infinite. Very good. Good job on that. So those of you who said, "no, that circles could not have the same centers" do you now see that there are possible ways that circles could share the same centers?

Mrs. Clark is not just praising the students. They know that they have given powerful answers and have become confident in taking risks in voicing their thinking. The students eagerly shared their answers. The students are excited about their ideas and their solutions. Several different examples of how more than one circle could have the same center have been presented here. Notice that none of them came from the teacher. When presenting an argument for their answers, students have built up confidence in their thinking.

By taking the students' ideas and thinking seriously, Mrs. Clark has shown them that their ideas and thinking is valuable. By allowing them to explore the problem without making judgments, she has shown them that she values their thinking more than guessing about the answer that she thinks is correct. The students learned that if they voice their ideas, their ideas will be heard and valued. According to McLeod (1989), if students regularly have positive experiences with mathematics, they will develop attitudes of curiosity and enthusiasm. McLeod noted that two major goals of teaching should be helping students understand the value of their mathematical thinking and developing student confidence.

Builds Instruction on Students' Articulation of Their Thinking

Mrs. Clark *requires* students to participate actively and explain their thinking in order to increase their understanding. During whole-group instruction, Mrs. Clark asks questions that elicit the student's thinking. She does not ask questions that push students toward the right answer but questions that probe for their understanding. Fosnot (1989) wrote that teachers should ask students nonleading questions to uncover misunderstandings. Fosnot went on to write that "asking questions about thinking causes people to think more" (p. 88).

Mrs. Clark poses a problem for her students and then watches and listens to what they tell her and each other. She pays close attention to her students' learning and explanations and often builds her instruction around the students' answers. Mrs. Clark uses her influence as the *guide* in the class discussions to evaluate a students' answer for its potential toward building a productive discussion. She encourages, "Tell me more about that. I'd like to know more about what would happen if . . . ." She recycles the students' ideas to new problem situations. Remember in Chapter IV that she decided to use Hal's comment about the daily deposit to build the next day's lesson. He said that they (the class) would not reach $1 million before the end of the year.

Mrs. Clark often builds the lesson on the students' explanations of their thinking. The following lesson about probability demonstrates how she builds a lesson that ends with students presenting and discussing mathematical concepts that are complex for many secondary students to understand. The problem is

> In a sack of 30 coins, the probability of drawing out pennies is 1 out of 2. A nickel has a 2 out of 5 probability of being drawn. A dime has 1 out of 10 chance of being drawn.
>
> There are ____ pennies, ____ nickels, and ____ dimes in the sack.

Mrs. Clark:	How many pennies do you think are in the sack, John?
John:	15 pennies.
Mrs. Clark:	Why do you think 15?
John:	Because half of 30 is 15.

Mrs. Clark later said that she was surprised that John gave the correct answer so quickly. This year is the first time she has formally taught probability to fourth graders, so she was uncertain about where their knowledge would begin. After John's answer, the students discuss more about why there might be 15 pennies in the sack. Using John's answer, Mrs. Clark's writes on the chalkboard 1/2 and 15/30 beside each other.

$$\frac{1}{2} \qquad \frac{15}{30}$$

She asks the students if they think any connection might be made between the two fractions. Jerry says he thinks they are equal. Others contribute their opinions. Mrs. Clark finally puts = between the two fractions (1/2 = 15/30). Mrs. Clark records their ideas and provides a representation of their reasoning to help develop students' understanding of the written symbols. Lampert (1989) suggested that teachers should provide visual representations and symbols for students' answers. She also writes 15 pennies in the blank in the question. They begin a discussion about how many dimes are in the sack. Many students have guesses. Most are incorrect. Eddie says that the 1 out of 10 probability should be 1/10. Mrs. Clark writes this on the board.

$$\frac{1}{10}$$

Suddenly Jim raises his hand and says excitedly,

Jim: Could you do the same thing (the same thing they did to get 1/2 = 15/30) to that (he points to the 1/10) and see what numbers go into the top and bottom numbers?

Mrs. Clark: All right. We are going to do the same thing here. What is the denominator going to be?

Some students: 30.

Mrs. Clark: What is the numerator going to be?

Ken: 20. (There were no other suggestions from students. So Mrs. Clark writes 20/30 on the chalkboard.)

Even though Mrs. Clark knows this answer is incorrect, she writes 20/30 on the chalkboard.

Mrs. Clark: Dan, can you help us out?

Dan: Ten times 3 is 30. So it would have to be 1 times 3 is 3. (Mrs. Clark writes $\frac{1}{10} \times \frac{3}{3} = \frac{3}{30}$ on the chalkboard.)

Mrs. Clark: Okay. Did everybody see what Dan did? Ten times 3 is 30 and 1 times 3 is 3. What does this tell us?

Hal: That 3/30 [are] 1/10.

Mrs. Clark: What else does it tell us? Dan?

Dan: That there is [sic] 3 dimes.

Mrs. Clark: All right. Dan is saying that there are going to be 3 dimes.

Most of the students agree with Dan, so Mrs. Clark writes 3 in the blank for dimes on the problem. With several students' contributions, Mrs. Clark builds with nickels, in the same way as with the dimes, the new ratio of 2/5 = 12/30. However, some disagreement develops. Joan provides a reason why she does not think that 30 should be in the denominator for the fraction.

Joan: If we are at 2/5, wouldn't you start with the denominator at 15 because 15 of the coins are already gone?

Mrs. Clark: That's a good question. Joan is suggesting that the denominator should be changed to 15 because we have decided that there are 15 pennies already gone.

A Student: Say it again.

Mrs. Clark: What Joan thinks is that the denominator should be 15.
Her question was--since we already used 15 pennies, shouldn't there only be 15 coins left?

Hal: No!!! (He is adamant with his statement.)

Mrs. Clark: Hal is saying no to you, Joan. Why do you say, no, Hal?

Hal: Because the (does not remember what term to use) top number

Mrs. Clark: Numerator

Hal: Of 2/5 is 2. The numerator of 1/10 is 1, so 3/30. So 12/30. If you add 12 and 3, it equals 15 and that is the other half for the 15 (pennies). 15 + 15 would equal 30.

Mrs. Clark: Hal is saying that we need all the same denominators. When we add the numerators--we have 15 + 3 + 12 and that is 30. (Mrs. Clark writes the following on the chalkboard.)

$$15 + 3 + 12 = 30$$

Mrs. Clark took John's answer of 15 and his explanation that half of 30 is 15 and built this day's lesson around it. If they had been discussing conditional probability, Joan was actually correct in her thinking about the probability of drawing out nickels. If they had actually taken out the 15 pennies, there would only be 15 coins left. Mrs. Clark said later that she was very pleased with what happened. Their thinking about probability was deeper than she had expected. She said, "I think they know probability intuitively. The reason they began to have trouble later is that they are required to think about it in certain procedural ways that are not intuitive for them." She was very excited by Joan's thinking. She said, "That was kind of neat. She was really thinking. That was so important to me."

Looking back at this lesson, we can see how Mrs. Clark guided students toward conceptual understanding of probability. The lesson helped them construct concepts about ratio and proportion. The discussion showed how she built upon the explanation of one student, John, in knowing that 1/2 should be 15, to lead the discussion to complex concepts for fourth graders. As Mrs. Clark recorded their ideas and provided a representation of their reasoning, she gave her students the opportunity to connect their conceptual knowledge to the procedures (NCTM, 1991). The students listened closely trying to understand their classmates' thinking. Several times during the lesson, students asked to have responses repeated. At the end of the discussion, Hal gave an interesting comment, "We are getting smarter everyday, Mrs. Clark."

Asks Questions to Create Cognitive Conflict

Mrs. Clark creates cognitive conflict for her students by asking questions to help students rethink their answers or resolve mathematical misconceptions. Mrs. Clark's questions are not designed to direct students to thinking narrowly toward the answer she is looking for but are designed to help students experience conflict with their inaccurate thinking so that they can accommodate correct new knowledge. Mrs. Clark causes the students to rethink, not only to get them to change their minds about answers, but in order for them to understand *why* their answers are inaccurate and will not fit within a new conceptual framework. Piaget (1977) called this conflict, *disequilibrium.* Disequilibrium happens when a student's current knowledge or cognitions do not help the student explain the new situation. Simon and Schifter (1991) explained that construction of new understanding is elevated when students experience disequilibrium and must mentally modify their present knowledge to assimilate their new experiences. The writers added that "disequilibrium leads to mental activity and a modification of previously held ideas to account for the new experience" (p. 310).

During discussions, as students provide answers to problems and come up with solutions, Mrs. Clark often acts as moderator to help students evaluate the correctness of their answers. If the students have developed a pattern during their work, such as "when you divide you always get a smaller number," at this point, Mrs. Clark might say, "You've given me some examples when you are correct. How about this one?" Brooks and Brooks (1993) agreed with this approach by suggesting that "constructivist teachers engage students in experiences that might engender contradictions to students' current [thinking]" (p. 112).

During discussions, students hear about and see specific ways that might help them refine their thinking. Mrs. Clark asks students to paraphrase other students' ideas. She encourages students to examine their mathematical strategies. She allows the students to resolve their errors to help them construct new problem-solving strategies. One technique that Mrs. Clark uses when students answers show errors in their thinking is to help the students make up their own questions that help them to rethink their solutions. According to Fosnot (1989), in order to maximize students' understanding, it is important to probe students' thinking to cause them to ask questions of themselves.

Mrs. Clark asks questions that probe the students' understandings. With her questions, she guides the students to focus on what makes their thinking problematic. With her questions she helps the students to understand their misconceptions and possible trouble areas for their future thinking. She says,

> I try to think where they go with their thinking. Sometimes I can help them back their thinking up and get them to realize they missed this point or they interpreted something wrong. Sometimes I go to other students that might communicate their responses. I might say, "Let's look at Jimmy's point in this. Would anybody like to comment on that?" Well students might say, "Well, I think it wouldn't be a triangle because . . . ." Jimmy might have given an invalid response, but he had a reason for it. Sometimes I might say, "Come up here and show me."

As all previous excerpts from the classroom discourse illustrate, Mrs. Clark listens and continually asks the student to explain and justify their answers. "Why do you think that?" or "How did you get your answer?" is a standard question, asked about correct answers as well as about incorrect ones. The following is a sample discussion about an incorrect answer.

The students have already solved a problem that involves a square that is 4 centimeters wide and 4 centimeter long. They have decided that the area is 16 square centimeters and that the perimeter is 16 centimeters. As an extension to this problem, Mrs. Clark begins the following discussion.

> Mrs. Clark: Are the perimeter and area always the same for a square?" (No responses.)

She repeats the question and asks it a little differently.

> Mrs. Clark: Is the perimeter always going to equal the area for a square? It did on this square. Is that going to be true for all squares?"

Mrs. Clark draws a diagram (see Figure 19). By the time that she completes the diagram, many students have their hands up.

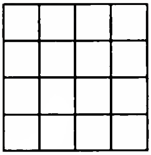


Figure 19. A Square.

Mrs. Clark: Kim, is that going to be true for all squares?

Kim: Yes, I think so.

At this point, Mrs. Clark does not know whether Kim is answering incorrectly because she does not understand the contrast between perimeter and area, because she might be answering a different question from the one being asked, or because she is making a guess. Consequently, Mrs. Clark asks Kim to explain her reasoning. When Kim does this, both teacher and some students know where the inaccuracy originates.

Mrs. Clark: Can you tell me why?

Kim: The sides are all 4. There is 4 on the one side and 4 on the bottom and there is 4 on the top and 4 on the other side. So 4 + 4 + 4 + 4 = 16. That is the perimeter. Then you times the number of centimeters of the side by how many sides there are. So 4 times 4 is 16.

Mrs. Clark: So, that's why you think that the perimeter and the area for all squares are the same?

Kim: Yes.

Some students agree with Kim that squares will have equal perimeter and area. However, many students have their hands up to disagree with how Kim found the area. Some of them explain that her answer is incorrect because she used the same method for finding both the perimeter and the area. After this discussion, Mrs. Clark asks, "Jill, what do you think?"

Jill: No, because when you multiply, it might be a 5 instead of a 4. When you multiply 5 times 5, you get 25. When you add 5 four times, you get 20.

Notice that this explanation did not come from the teacher but from another student. Mrs. Clark then asks another student to explain how Jill found her answer.

Mrs. Clark: She gave an example of one that is not. She used a square whose side is 5. When she found the area she said what? Jerry?

Jerry: 5 X 5.

Mrs. Clark: And got what?

Jerry: 25.

Mrs. Clark: How did she get the perimeter? John?

John: 5 + 5 + 5 + 5.

Mrs. Clark: What did she get for the perimeter?

John: 20.

Mrs. Clark: Are 20 and 25 the same?

Mrs. Clark asks questions that allow students to reveal their understanding of perimeter and area. She also uses Jill's correct answer to create cognitive conflict within the students who answered that the perimeter and area are always the same for a square. The students experience *disequilibrium* and, to resolve the conflict, are trying to find examples of squares that will have congruent perimeters and areas. Michael is convinced that a square with the sides of 1 centimeter will have equal area and perimeter. But another student tells him how 1 + 1 + 1 + 1 = 4 and 1 X 1 is 1. After about 10 minutes of exploring other examples, students do not find another example where the area and perimeter will be the same for a square. They finally decide that the perimeter and area are *not* always going to be the same for a square.

Before Mrs. Clark asked this problem, many students had superficial views of perimeter and area. Some did not distinguish the difference between perimeter and area.

Some students seemed to have a more procedural view and had not learned the meaning of the factors of length and width in the area algorithm. Because she knew that other students were thinking in ways similar to Kim, Mrs. Clark built her lesson around Kim's error in thinking. Instead of showing the students the right answer, Mrs. Clark asked for other students' reactions. Instead of explaining Jill's very thorough and articulate answer, Mrs. Clark asked other students to provide specifics of Jill's thinking to see if they have understood what Jill has said. The students' contributions helped clarify and justify Jill's solution. Through the discussion of this problem, all students had the opportunity to evaluate their thinking, find errors, and change any misconceptions. After having an opportunity to see the mathematical evidence, Kim and other students with misconceptions changed their minds. Mrs. Clark says, "When they come to an error in their thinking or an error in their justification or a misconcept, they can correct it. That usually puts them right back onto the path of coming to a correct solution." When this topic came up again, the students' answers demonstrated that they had a deeper conceptual understanding of area as the number of square units inside the figure and perimeter as the distance around the figure. Cobb (1994) agreed that when students reflect upon their disequilibrium, they modify their own concepts.

Labinowicz (1985) noted that students' errors are actually natural steps to understanding. When one child makes an error, a unique learning opportunity arises when another child attempts to assist the first child in clarifying his or her thinking. Student's errors are necessary steps toward the reconstruction of ideas at a higher level of understanding. Labinowicz suggested that if teachers deny children the right to be wrong, then they are denying the complexity and interrelatedness of mathematical ideas. He wrote that teachers need to listen effectively and collect information in order to adapt their teaching to the students' thinking. He further added that when students answers demonstrate errors in thinking, teachers can either interfere with students' ability to construct correct knowledge by just saying that the answer is incorrect and telling students the correct answer, or teachers can help students build connections from their present understanding to the new understanding they need to change their thinking.

Builds on Students' Prior Knowledge

Mrs. Clark guides students to connect their past learning experiences to their present and future learning. She values the students' prior knowledge. Mrs. Clark provides experiences for students in class that relate to their lives outside of mathematics class. She stimulates their interest so that they actively involve their minds in making sense of their past learning experiences. Kamii (1988) noted that "relationships cannot be put into children's heads from sources external to them" (p. 51).

Mrs. Clark believes that students have valuable intuitive, informal knowledge that is important for them in understanding mathematics. Webster's Dictionary defines intuitive knowledge as immediately knowing something without conscious reasoning. Mrs. Clark builds on students' prior knowledge so that they will not simply memorize procedures for solving problems that they do not understand. Kamii (1989) suggested that "if we encourage [students] to develop their own ways of thinking rather than requiring them to memorize rules that do not make sense to them, children develop a better cognitive foundation [for the procedures]" (p. 14). Students need to analyze their thinking in depth to make connections and relationships (Fosnot, 1989).

Mrs. Clark understands and diagnoses the students' current levels of thinking so that she can organize instruction that will encourage students to make connections from prior knowledge to new knowledge. Mrs. Clark observes her students solve problems and listens to their strategies so that she can obtain information about their prior conceptual understanding. She then builds her instruction upon the students' prior conceptual knowledge. She says, "I need to find *where* they are building knowledge. I am hoping to add blocks to their building. It may be at different levels. They come to me with a certain amount of knowledge. I am hoping to add to that."

When Mrs. Clark began teaching probability, she involved students in real experiences learning about chance. One activity in which they participated centered around the question--Is this a fair game? Students worked together in cooperative pairs as they played a game in which they threw two die. They kept a record of throws to see which or if certain sums of the die came up more often. During these activities, she sought to understand the students' intuitive level of knowledge about probability. Van de Walle (1993) suggested that knowing student's intuitive knowledge is important before students learn algorithms.

In the probability problem that I presented earlier, Mrs. Clark used the student's intuitive knowledge of probability to scaffold the students to develop the procedures for writing ratios for probability. Bruner (1986) used the word "scaffolding" as the process by which a competent adult helps those less competent gain a higher level of thinking. According to Lampert (1991), "beginning a problem in a familiar domain . . . is a way that teacher and students can communicate in what Vygotsky (1962) called the 'zone of proximal development'" (Lampert, 1991, p. 127).

Mrs. Clark helps students develop their own procedural algorithms. She guides lessons in such a way that the students learn the procedures in a conceptual context. According to Bruner (1986), when a teacher helps students learn to do "*with* her what they could not do *without* her," they are in the zone of proximal development. The following example of a lesson on fractions shows how students are *inventing algorithms*--connecting procedures to the concepts. They are inventing the procedure for changing mixed numbers to improper fractions.

Before I describe this example, I will give a background of what students had been doing with improper fractions. The students had been working for several days with fraction circles. The students explored with these materials to model their conceptual thinking. Lampert (1989) wrote that being able to represent concepts through using materials contributes to long-term understanding. Multiple representation of concepts help students make sense of mathematics. Students in Mrs. Clark's class worked in cooperative groups of five using fraction circles to represent unit fractions, such as 1/5, and nonunit fractions, such as 2/3. They had discussed equivalent fractions using the fraction circles and fraction strips. On the day previous to the following lesson, students modeled, with fraction circles, how mixed numbers were also improper fractions and the reverse. In the modeling lesson, Mrs. Clark put the students in cooperative groups and asked them to represent 3 1/2 circles by showing any combination of fractional units. Some groups put out 7 half circles (see Figure 20). Another group laid out one circle with 2 half circles, one circle with 3 circles that showed thirds, one circle that showed 4 fourths, and finally 1 half of a circle (see Figure 21).

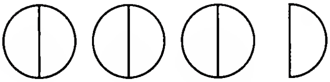


Figure 20. Half Circles

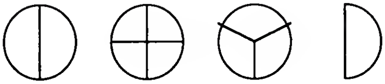


Figure 21. Mixed Circles

Mrs. Clark helped students understand that 3 1/2 was the same as 7/2. This lesson lasted about 45 minutes. By the end, most students were competent in representing mixed numbers and improper fractions.

The next day to put what they had learned about mixed numbers and improper fractions into real-life context, Mrs. Clark asked the students to measure different items in feet and yards. They came up with an amount of 3 1/3 yards. They decided that 3 1/3 yards also equaled 10 feet. Mrs. Clark wrote on the chalkboard 10 feet = 3 1/3 yards. She asked, “What is another way of writing 3 1/3 yards?

Dan: 10/3.

Mrs. Clark: 10/3. Now, I want you to look at something very carefully. I am going to write $3\frac{1}{3} = \frac{10}{3}$. Would you see if you can tell me where I can get 10 and 3 from?

Students: Oh. Oh. (many hands are up)

Mrs. Clark: Eddie, tell me where you think I can get it from?

Eddie: That 3 feet and 3 feet and 3 feet is 9 feet and 1 [foot] more is 10 feet.

Mrs. Clark: Now, look at $3\frac{1}{3} = \frac{10}{3}$. Where did your threes come from? Come and point to the 3 that you added.

Eddie: (Eddie pointed to the whole number 3 in 3 1/3.)

Mrs. Clark: So Eddie said, 3 + 3 + 3 + 1 is 10. Looking at this problem, where do you think he got the 1? Tony?

Tony: (Tony pointed to the 1 in 1/3.) There [are] 10 feet and 3 is on the bottom because there are 3 feet in a yard.

Mrs. Clark: Okay. Tony has explained where I got these two numbers. (As she talks she points to the numbers.) He said 10 is the number of feet we have today and 3 is the division that a foot makes in a yard. I want you to look at just this (points to $3\frac{1}{3} = \frac{10}{3}$). Can anyone else come up with a way to get 10/3 from 3 1/3?

Missy: (Does the algorithm) 3 X 3 is 9 + 1 is 10.

Mrs. Clark: Missy, come up and point to the numbers as you say them. (Missy comes up to the chalkboard.) Where does your first 3 come from?

Missy: The bottom. (She points to the 3 in 1/3.)

Mrs. Clark: The denominator. (Missy then points to the whole 3 and then the 1 in the numerator of 1/3.) Let’s take a look at another one. What do you think the improper fraction would be 2 2/3? (The students work the problem in their math logs. Several hands go up.) Jerry?

Jerry: 8.

Mrs. Clark: 8 what?

Jerry: 8/3.

Mrs. Clark: Jerry, tell me what you did?

Jerry: I said 2 X 3 = 6 and then I added 2 to this and I got 8.

They do several more problems like these. Finally, Mrs. Clark says, "All right, you know we are going to keep looking at our measurement tape (*The Standards* section during February) and see if this is what we come up with every time."

Mrs. Clark helped students develop the algorithm for changing mixed numbers to improper fractions by leading them from the familiar conceptual territory of their measuring problems and experience with fraction circles and strips to symbolism and algorithms. She helped students make connections from the concrete to the abstract by supporting them with materials and questions while they invented their own ways to solve problems. Mrs. Clark observed their models to assess their conceptual knowledge and then helped them connect this knowledge to the procedures. She did not *tell* them how to do the algorithm. She said, "Most of the time when children understand the concept, they can come up with the algorithm." Kamii (1989) noted that when children invent their own algorithms, they become more competent than when the algorithms are told to them. She added that "procedures children invent are rooted in the depth of their intuition and their natural ways of thinking" (p. 14).

According to von Glasersfeld (1990), students' understandings of concepts are extremely important for building a knowledge base on which they can construct new knowledge. Understanding mathematics is distinguished as a process of constructing concepts and actively interpreting the connections. This is the essence of constructivism, encouraging students to construct their own knowledge and achieve intellectual autonomy, not imposing knowledge (such as algorithms) onto them (Kamii, 1989).

With this lesson as with many others, Mrs. Clark has helped students make the connections from their prior knowledge to new knowledge. Piaget (1973) wrote that "knowledge is . . . changing. . . . It is not momentary; it is not static. . . . It is a process of continual construction and reorganization" (pp. 1-2). Later, in other lessons, the students exhibited evidence that their understanding of fractions and mixed numbers continued to grow as they confronted new problem situations. Their work demonstrated that they had learned the algorithm connected to the concept. They made very good scores on this portion of their CTBS exam although they never practiced through repeated drill. Hiebert and Carpenter (1992) noted that when students internally make sense of the concepts and then make appropriate connections with the external representations, the understanding has occurred.

Helps Students Construct a Knowledge Base

Continually, Mrs. Clark provides opportunities for students to construct a knowledge base of concepts, mathematical vocabulary, and symbols before they will *officially* be learning them. Piaget (1973) wrote that the growth of knowledge is the result of individual constructions made by the learner. One of the ways that Mrs. Clark helps her students construct a knowledge base for thinking about mathematics is by helping them learn and use the "language of mathematics."

Mrs. Clark thinks that vocabulary in mathematics is important. She teaches students how to think and speak mathematically. She provides correct mathematical terms. Duckworth (1987) suggested that we give children "language tools" not only to facilitate clearer thinking but also to help them to communicate their thinking to others. When children use correct terms, we can pay more attention to *what* is being said rather than *how* it is being said. Mrs. Clark encourages the students to say the correct words. An example is a previous lesson involving probability. She provides Hal with the term numerator and writes probability as fractions and proportions. She introduces her students to new words in a nonthreatening way. She does not require that they take vocabulary tests, but over time she tries to get them to use the language of mathematics. When I asked why she emphasized vocabulary in her teaching, she responded,

> I think hearing big words used regularly makes them less afraid of them when they have to know them. We don't really get to the point of studying terms like *90*

> *degree angles* in the fourth grade. But if I can slip that in every now and then, when they have to know them in later grades, it's not a foreign idea. [For instance] when I get to fractions, I don't have to teach what is a numerator and denominator because I've already used it and used it in a way they know that they are not going to be tested. They have some prior knowledge with it.

Mrs. Clark usually presents new vocabulary words in a context in which the students are studying something familiar. When she tells them new vocabulary words, however, she is not just telling students answers to problems. Students learn the vocabulary through using it. She helps them build the mathematical terminology to go with thinking they already possess. Lampert (1990) suggested it is the role of the teacher to help students acquire technical knowledge and skills in the discipline of mathematics. By "acquiring these tools (language and symbols), the individual is able to articulate the meaning of his or her ideas and construct more sophisticated understanding" (Lampert, 1990, p. 133). The students need not *know* the vocabulary words for months after they are exposed to the words. For example, by the time students were involved in regular work on fractions, they had investigated and applied fractions in many problem-solving situations and learned about numerators and denominators. One morning Mrs. Clark made the decision to introduce them to the term *90-degree turn.* She did this during the discussion of the problem of the day. She asked the following question: Which figures are the same (see Figure 22)?

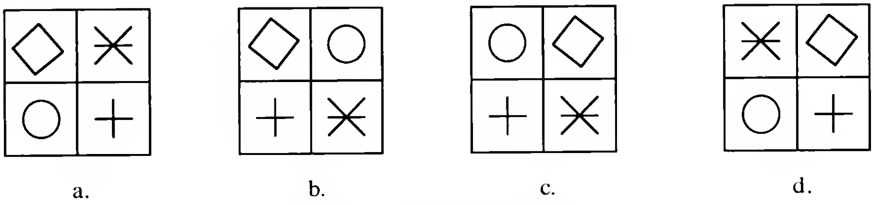


Figure 22. Changing Positions

What would you do to one of these to get them in the same position?

George: (a) and (c) are the same.

Mrs. Clark: Why did you pick (a) and (c), George?

George: (Going up to the drawing) I took (a) and turned it this way (moves hand to the right), then I saw that that was like (c).

Mrs. Clark: You are saying you turned it this way. What is that called when you turn it this way?

George: Rotate.

Mrs. Clark: Do I turn it one position, two positions, three positions? Tell me more about how far you rotated it. Beth?

Beth: (Beth had made the first figure and then cut it out. She showed how she turned one turn.)

Mrs. Clark: Which direction did you rotate?

Beth: Right.

Mrs. Clark: To the right. There is another way of saying that. We can say that we make a *90-degree* turn. When you are looking at an angle, from here to here (she pointed from a northerly direction to the east) is 90 degrees. So, if I flip it one time, I make a 90-degree turn. If I flipped it twice, I make a 90 + 90 or

Mrs. Clark is drawing this diagram (see Figure 23) as they talk.

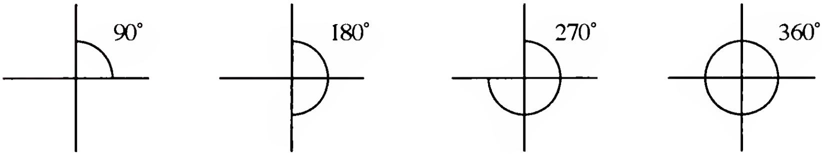


Figure 23. Quadrants

Jim: (Waving his hand, he calls out) 180 degrees.

Mrs. Clark: Okay. One more turn?

Jim: 270.

Mrs. Clark: Or if I turned it all the way around back to where I started?

Jim and Hal: 360.

Most likely the students had not heard the term *90-degree turn* before. Mrs. Clark later said that this was the first time she had presented this idea to them. Most certainly, they had not seen a rectangular coordinate axis before this lesson. She said that they would not be required to know about a 90-degree angle in the fourth grade. But, at least, when they did have to know what it meant, they would have some prior knowledge. After this lesson, Mrs. Clark and the students used this term for the remainder of the semester.

The previous excerpt is just one example of Mrs. Clark's attempt to help the students build their knowledge of mathematics. In the past probability lesson she also used the symbols and procedures for writing their thinking about ratios and proportions. She uses equal marks to show their thinking about equivalent ideas. If we look back at the “counting tape” example in Chapter IV, we can see that she showed the students how their thinking about divisibility could be written through using the division algorithm. She used symbols on the counting tape and symbols on the calendar. She used number sentences to represent their solution. She used symbols to help the students visualize the concepts. She used fractions to show division and probability. By using mathematical symbols, Mrs. Clark helped students learn another language by which to communicate their thinking.

Mrs. Clark reinforced through oral language and written symbols that mathematics is a discipline that has its own unique vocabulary and symbols. By learning to speak and write mathematically, students came to know mathematics. Lampert (1990) found in her research that students need to learn mathematical language as tools for communicating. Fennema, Carpenter, and Peterson (1989) noted that to help children make connections from concepts to procedures for writing symbols and algorithms, teachers must introduce ways to represent the knowledge that children have already acquired. The knowledge that Mrs. Clark helped the students construct is important for them in understanding that mathematics is about the relationships of procedures to concepts.

Encourages Reflection and Helps Students Extend Their Thinking

Even though Mrs. Clark seeks to match the content of the curriculum to the students' cognitive abilities, she continually encourages them to extend their thinking. To extend their thinking, Mrs. Clark does not always direct the students' attention toward an incorrect answer nor does she design her instruction to simply direct students to the correct answer. She encourages the students to look more deeply at the problem itself to explore their understanding. She helps students extend their thinking to a variety of solution strategies. She supports them to apply their learning to new situations. She often avoids comment about whether an answer is correct. If she thinks they are not ready to consider some aspect of the problem or some new information, she proceeds with the lesson. However, because of her efforts to get students to evaluate their thinking constantly, they have developed reflection as a habit even when she goes on with some new portion of the lesson. According to Duckworth (1987), it is by thinking and reflecting about their thinking that students get better at thinking. Duckworth added that if students do the work themselves, reflecting and making the connections, it is more likely that students will be able to repeat what they have learned. Cobb (1994) called this type of restructuring thinking *reflective abstraction*. He noted that reflective abstraction helps students consider their actions to project their concrete experiences to abstract mental representations.

Evidence that students appear to have developed habits of reflecting are demonstrated in the following episode. At the beginning of this lesson, students had constructed lines, line segments, and rays with geoboards and dot paper. Mrs. Clark had asked them the day before to bring in examples of lines, line segments, and rays from home. Through much discussion and debate, the teacher and the students had developed definitions for lines, line segments, and rays. The students had finally decided that if they put arrows on the ends of line segments, all line segments can be made into lines. These arrows showed that the lines are infinite (the definition that Mrs. Clark has given them).

Mrs. Clark then asks the students to give examples in the classroom of intersecting lines. One student names some of the letters of the alphabet. Mrs. Clark takes this opportunity to move the lesson in that direction. She asks them to look only at the capital letters of the alphabet. She then asks which of the letters have intersecting lines. Several students write their letters on the chalkboard. They name many letters--T, L, M, and so on. Finally, one student, Ann, suggests R. Mrs. Clark asks Ann to come up and make her R. She makes it like this, . After a short discussion the students decide that these lines do not intersect. Jim says, "But if I make an R like this, , (he writes on the chalkboard.) they do intersect." Everyone agrees. Mrs. Clark asks, "What do you think about the first R, ( )?" No one answers. Mrs. Clark is not sure if they are ready to reconsider if these lines intersected, so she continues on without commenting on the R to allow another student to suggest the next letter with intersecting lines. Even though students had previously discussed lines and their characteristics, the students had only *heard* that lines are infinite. They had not demonstrated that lines can be extended to show infinity. After about 10 minutes, Dan says, "Mrs. Clark, Mrs. Clark, I have an idea. Remember, you said that a line goes on forever. (He hurries to the chalkboard.) If we make this line (referring to Ann's R) go farther, it will cross this line." He draws on the chalkboard, . Mrs. Clark is obviously pleased. She claps her hands and says,

> Yes, in geometry, a line is a set of points that go on forever. They keep going in the same direction. Very good, Dan. I was hoping someone would see that. That's one of the best ideas I've seen in a long time.

Mrs. Clark's question had helped Dan to extend his thinking. The next day she told me that after the lesson the students kept talking about lines going on forever and trying to find examples that would be good to use.

In this lesson students got the opportunity to analyze and reflect about intersecting lines and to reason mathematically. As their knowledge of certain mathematical ideas, like infinity in space, became familiar, they learned to use this idea in new situations. Dan demonstrated that sometimes reflection takes time. After thinking and reflecting on the problem, Dan conceived of a diagram that he could draw that would show his thinking about intersecting lines. Some teachers might say that Dan had not been paying attention to the other students' new ideas while he had been thinking about the old one, but Mrs. Clark has shown her students that it is all right to reflect upon the activities in which they are involved and come up with contributions at a later time.

Mrs. Clark encourages students to push their thinking to higher levels. Students learn to analyze and synthesize their knowledge so that they learn that mathematics is about relationships that grow to a *whole picture*. Mrs. Clark provides opportunities for students to learn to extend knowledge to new situations. Dan's making the connection between intersecting lines and lines "going on forever" shows that he has analyzed and pushed his thinking to a higher level of synthesizing. Mrs. Clark helps her students come to understand that "thinking mathematically" is a developmental process in which they build and extend their learning over time. When they construct new conceptual knowledge, they learn to reflect upon their old knowledge and use it in new ways. According to Brooks and Brooks (1993), if teachers do not know how to teach so that students can learn to extend their thinking, they will not know what ideas are within students' reach. It is important that teachers select mathematical problem-solving situations that help students reorganize their concepts (Schifter & Fosnot, 1993).

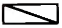
Teaches the Students to Take Responsibility for Their Learning

Mrs. Clark guides the students in proving the correctness of their own answers and does not establish herself as the authority for correct answers. The students collaborate with the teacher in the discussions and interactions so that the responsibility for learning becomes theirs. Kamii (1988) suggested that the teacher should be careful not to become the omniscient authority who decides right and wrong answers. The teacher should present another point of view on an equal footing. Mrs. Clark helps students initiate and achieve responsibility for learning. She provides them with the opportunities to construct mathematical concepts for themselves.

In Mrs. Clark's class, the students demonstrated autonomous behaviors by often evaluating their own learning. They say, "We are getting smarter everyday" or "I came up with a good one." According to Piaget (1973), autonomy should be the goal of education. Piaget's theory of autonomy is that a person can make decisions by taking relevant factors into account and thinking independently (Kamii, 1989). As the school year progressed, the students became independent thinkers and looked less often to Mrs. Clark as their authority for deciding if their answers were correct. Many students seemed to accept themselves as having the primary responsibilities for learning and talking about their learning. Kamii (1989) noted that children develop autonomy when they exchange points of view with other children.

Over time students seemed to become secure and confident about their thinking and more willing to argue for the legitimacy of their answers. A good example of their independence was their math logs. While thinking about and solving the problems, the students did not run to the teacher to find out if they were working the problems correctly.

Students often initiated their own learning without waiting for the teacher to call on them. They involved themselves in the discussions without being prompted. The students showed a willingness to decide on their own solution strategies for solving problems. They cooperated in helping others understand what they know. Mrs. Clark encouraged the students to find their own problems. An excellent example of a student who has become involved with the mathematics and learning to think on his own is the following example of Eddie.

For several days students had been studying symmetry. They had drawn figures on grid paper and used their geoboards to construct figures. They had drawn or modeled the lines of symmetry for these figures. One day students found lines of symmetry of triangles, squares, and rectangles. Some students suggested that an equilateral triangle had three lines of symmetry. A square had four lines of symmetry. After a long discussion, they finally agreed that a rectangle has only two lines of symmetry. (Many students thought that the diagonal of a rectangle would also be a line of symmetry, ). Students also found other objects in the room and discussed their lines of symmetry.

At lunch that day Eddie told some of the other students about a hypothesis that he proposed about lines of symmetry. Several students called to Mrs. Clark saying, “Come hear Eddie’s theory.” They next day she said to me, “Ask Eddie what he has discovered?” In the following explanation, Eddie told me how he developed his hypothesis about lines of symmetry and figured out a way to prove if his hypothesis was correct. He also explained *why* he thought his theory was correct.

Eddie: If a triangle has 3 equal sides, then it has 3 lines of symmetry. If it is a plane rectangular figure, like a square that has 4 equal sides, then 4 lines of symmetry. A pentagon has 5 equal sides, then 5 lines of symmetry. Same thing on the dodecagon. Make a star so ten equal sides, then 10 lines of symmetry in it. I don’t know about a circle. If you take little pieces of it like on a geoboard, it might have many lines of symmetry, but maybe none.

Mrs. Steele: How did you first think about this theory? How did you prove it?

Eddie: I remembered some of the problems and the calendar. I found 2 on the rectangle and then the square when Mrs. Clark put it down there. I made the lines of symmetry, so I had 4 because 2 sets of equal sides. I saw that on the square. I found zero lines of symmetry if there are not any equal sides. After I saw the square and the rectangle, I saw others.

Looking back, we can see that Eddie has formed a hypothesis, devised a plan, and developed a proof for that plan. He has become confident in his ability to think for himself and has become so sure of his thinking that he tells other students about it at lunch. He does this without first going to Mrs. Clark to find out if his hypothesis is correct. Eddie does not need her to validate his thinking before her shared it with the other students. This example provides evidence to support that he has become an autonomous and a mathematical thinker. By helping students learn to assess their own learning, Mrs. Clark has helped them to achieve greater responsibility for their learning.

Obviously, Mrs. Clark has helped her students to become a mathematical community. Students have learned to hypothesize and to justify their ideas. She has helped them learn that mathematics is a systematic process of coming up with an idea and seeing if it works. According to Lampert (1990), mathematics is about coming up with a theory and testing it out. When students work in an environment where they can make their own proofs about their reasonableness of their ideas, then knowing mathematics in school becomes closer to what it means to know mathematics within the discipline. By learning how to prove theories, students learn how truth is established in mathematics.

When I asked Mrs. Clark, “What do you expect from your students?” She answered,

> The most obvious thing is to gain knowledge and understanding of whatever it is I am trying to teach. I want them to grow in what they know and what they understand. I want them to be able to apply their learning in other things. Within a lesson, students should interact with their complete thinking--be a part of trying to arrive at some closure about what we are doing--whether it is an assignment in the book or whatever. They need to make sure that I know that they know--whatever way that I ask it--whether it is orally or showing me on a geoboard. Whether it is writing the answer down on a worksheet or doing a page in the book. Whatever way that I am seeking to find out if they understand. They should listen carefully to me and to other students, pay attention, and interact so that they can learn the knowledge and realize where they are at knowing it. If they are strong in understanding some concept, I think that they should try to help others to understand.

Mrs. Clark assumes the responsibility of creating an educational environment that encourages her students to accept the responsibility for their learning. She does this by encouraging the students to initiate their own inquiry or thinking. Brooks and Brooks (1993) noted that "students who frame questions and issues and go about answering and analyzing them take responsibility for their own learning and become problem solvers, and perhaps more importantly, become problem finders” (p. 103). When they present their thinking about ideas that they have initiated, she shows she values these ideas by having students discuss them with the class, by using the ideas for planning future lessons, by showing a positive attitude, and by insuring that students feel confident and motivated.

Summary

I have spent the preceding part of this chapter breaking down Mrs. Clark's teaching into 10 different instructional strategies that highlight her teaching practice. I have connected these strategies to constructivist learning theory and demonstrated how Mrs. Clark helped students develop mathematical power. These strategies can serve as guides to other teachers who want to become "constructivist teachers." These strategies can also help teacher educators in developing programs for preservice and inservice teachers.

However, just as Mrs. Clark asked her students to look at *whole* primary concepts before finding the parts, I want you to look at the *whole* quality of Mrs. Clark's teaching. In order to understand the entire picture, you must integrate all the strategies. In other words, her model of constructivist teaching is the sum of the parts. Mrs. Clark did not consciously use these strategies. In order to develop them, I analyzed the subtleties of her teaching.

Thinking mathematically is not something that can be scheduled during a particular part of the day separate from the rest of the school day. Mrs. Clark consistently involved students in high-level discussions. She seized the moment during lessons when a student's contribution had the potential to lead to rich discussions. Green, Kantor, and Rogers (1991) characterized these events as *teachable moments*. Mrs. Clark pushed students to think way ahead of what they already knew. They learned to describe their thinking and become excited about mathematics. They built meaning through their actions and their interactions. Many of the problems that I have described to you are what Lampert (1990) called *good problems*. Lampert suggested that a good problem involves getting students to reveal and examine their assumptions about mathematics. With good problems, Mrs. Clark helped her students construct a way of thinking about mathematics that is rich with powerful conceptual connections. She communicated to students that mathematics makes sense. They saw that much of mathematics is learned by making connections with physical objects in the physical world. By reflecting upon their actions, they learned to generalize from the specifics of one problem to other problems. Polya (1981) quoted Lichtenberg in saying, "What you have been obligated to discover by yourself leaves a path in your mind which you can use again when the need arises" (p. 103).

CHAPTER VI

BARBARA CLARK'S GOALS AND THINKING ABOUT INSTRUCTION

In this chapter, I examine how Mrs. Clark's conceptions about mathematics and mathematics teaching and learning affect her thinking about instruction. I begin by discussing her goals for her students in learning mathematics and her conceptions of mathematics and mathematics teaching and learning. I describe the past and present dilemmas that Mrs. Clark has encountered as she uses a teaching approach that implements constructivist learning theory. Finally, you will read about the decisions that she makes about instruction. I describe her planning decisions, including her sources of problems, her choices of mathematical tasks, her choices of mathematical tools, her time allowances, her organization of students, and her assessment choices. At the end of this discussion, I describe a group of lessons that is illustrative of Mrs. Clark's thinking about instruction and curriculum.

Mrs. Clark's Conception of Mathematics Influences Her Goals

Mrs. Clark wants to provide opportunities for her students to learn to "think mathematically." Her goals for her students are influenced by her own conception about what mathematics is--a way of thinking. She says, "It's a way of thinking about numbers and shapes and concepts and their relationships, how it all fits together." She believes that thinking mathematically helps students think better in all areas. Therefore, she extends her main goal of thinking mathematically by saying, "I want my students to learn to reason out things for themselves and value their own thinking." Mrs. Clark believes that mathematics is a perfect vehicle by which students learn to think and value their thinking.

Doing and teaching mathematics gives Mrs. Clark a great deal of joy. Nevertheless, she does not think that pursuing mathematical knowledge is an end in itself.

She wants her students to not only learn to think and reason but also to be able to relate and apply their learning to everyday life. She says,

> I think they learn math by developing their reasoning skills. They need to be encouraged to think and explain and build confidence in their ability to think through situations. They need opportunities to have experiences with math. The experiences I give can be two kinds of things. They can be the real-life situations--like I need math in order to figure my batting average. I want them to connect math to their real life. But they also need experiences with [math] that improves their thinking.

Mrs. Clark thinks that learning about mathematics will help her students "keep a more open mind about mathematics down the road when it gets harder." Finally, Mrs. Clark says,

> I want them to look at the fun side of math, the neat side of math, the challenging side of math and understand that it does connect to a lot of things. It is valuable in their whole learning experience. It helps them just to be a complete person.

Because Mrs. Clark's conception of mathematics is that mathematics is a way of thinking about relationships between concepts and ideas, she designs her instruction to help her students achieve this understanding. Several researchers (Clark & Peterson, 1986; Brown, Cooney & Jones, 1990; Ernest, 1989; Thompson, 1984, 1992) have concluded that teachers' conceptions--beliefs and knowledge--affect their thinking about instruction and their decision-making. Teachers whose conceptions about mathematics lead them to believe in the constructivist approach to learning emphasize that students must internally construct new knowledge through accommodating old knowledge rather than absorb the knowledge that a teacher *transmits* to students. Changes in teaching that are advocated by many mathematics educators and suggested by *The Standards* (NCTM, 1989, 1991), make it extremely important that we understand the conceptions of teachers who are implementing constructivist learning theory in their classrooms.

Mrs. Clark's Conceptions about Mathematics and Mathematics Teaching and Learning

Ernest (1989) and Thompson (1984, 1992) have identified three conceptions that teachers may hold about mathematics. Ernest summarized these three views: (a) The problem-solving view defines mathematics as continually expanding and created by people as they see patterns and synthesize knowledge. Mathematics is a process of inquiry, not a finished product. (b) The Platonist view defines mathematics as a body of knowledge with established truths that should be discovered not created. Mathematics is a product. (c) The instrumentalist view defines mathematics as a collection of skills, rules, and facts that people need to memorize. Again, mathematics is a product. The conceptions that Ernest defined involve both an element of beliefs and knowledge. Even though *views* are usually defined as beliefs (Nespor, 1987; Thompson 1984, 1992), Ernest's definitions combined both beliefs and knowledge. Thompson (1992) noted that researchers should not look at teachers' mathematical beliefs in isolation from teachers' mathematics knowledge.

I came to understand Mrs. Clark's knowledge and beliefs about mathematics by observing her teaching practice and listening to her answers to my questions. On the *Mathematics Beliefs Scales* questionnaire developed by Fennema, Carpenter, and Peterson (1987), she strongly agreed that teachers should encourage children to find their own solutions to mathematics problems even if they are inefficient. She strongly believed that mathematics should be presented to children in such a way that they can discover relationships for themselves. She strongly agreed that allowing students to discuss their thinking helps them to make sense of mathematics. She did not believe that there was one right way to solve a problem.

Mrs. Clark's beliefs are in line with the problem-solving view of mathematics that Ernest (1989) defined. The problem-solving conception is compatible with the constructivist learning principle that learning is a process whereby individuals internally construct new knowledge from prior knowledge.

Even though Mrs. Clark never mentioned the words "constructivist learning," by observing her teach, I saw that constructivist principles of learning mathematics were present in her teaching. Mrs. Clark believes that students personally construct mathematics during classroom interactions when they explain and justify their thinking. In her dialogue she talks about how each student needs to build his or her own knowledge in individual ways based upon prior knowledge. She gives students opportunities to construct powerful and correct mathematical ideas and concepts.

Mrs. Clark shows her conception of mathematics as growing and dynamic through both her innovative methods of teaching and the questioning of her students. Her questions to students are often "What do you think might happen if we do it this way?" "Why do think this works?" Her conception of mathematics mirrors that of constructivist learning theorists. Piaget (1973) and Kamii (1985, 1989) proposed that students need to construct actively new knowledge internally as they accommodate and assimilate their new knowledge. Vygotsky (1978) proposed that people construct knowledge as they interact and discuss their thinking. Von Glasersfeld (1990) and Cobb (1994) advocated an integration of these two processes by suggesting that students' individual construction of new knowledge is facilitated by interaction with others in a like community.

Mrs. Clark tries to help the students come to understand that mathematics is not a collection of rules and procedures but is about making sense of relationships and connections of concepts to one another. She transmits a sense of what mathematics is and what knowing mathematics should be by the activities and problems that she chooses for the involvement of the students. Through her thinking and decision making, she helps students learn the procedures while they develop and refine their conceptual knowledge, Mrs. Clark's teaching exemplifies Skemp's (1978) idea of "relational understanding." When she teaches, she helps students focus on the process of thinking about connections between concepts and procedures. This focus demonstrates to her students her belief that mathematics is about thinking not about only getting the right answer. The process of constructing mathematics through the interactions among the students and with her became the content of her lessons. Lampert (1990, 1991) demonstrated this idea within her teaching. The discussions and arguments of the students became the content that students learned. Within the context of students' discussions, Lampert guided them to construct *correct* answers. Lakatos (1976) noted that mathematics is created through a process of contradiction while people argue, refine, and improve conjectures. Lakatos suggests that disagreement during the process of proving arguments is central to knowing mathematics.

Mrs. Clark also has depth of knowledge of subject matter. She demonstrates in her teaching practice and decision making a sophisticated knowledge of mathematics as a subject matter. She shows an awareness of important concepts within mathematics and how these concepts are related. She demonstrates within her teaching how mathematical principles are established. She is aware of important questions in the field and the ways that reasoning becomes valid. Through her teaching, Mrs. Clark demonstrates that mathematics content is continually created. She uses the language of mathematics and leads the students to do the same. Her subject matter knowledge is integrated and well organized. She carefully sequences lessons because she knows which concepts students need to understand before proceeding to others. She knows what to teach and how to teach. The depth of her knowledge of the subject matter of mathematics was evident within Mrs. Clark's assessment. Her knowledge of mathematics helped her create better questions and problems for the students to reflect on and answer. Mrs. Clark says, "It is helpful . . . to understand the concepts that you are teaching. This [conceptual knowledge] affects the students' learning."

Shulman (1986) stated that subject matter knowledge, pedagogical content knowledge, and curricular knowledge are necessary for teaching. Mrs. Clark demonstrates knowledge of mathematics pedagogy. She knows how children's informal mathematical knowledge can help them build new knowledge. Her teaching practice illustrates her knowledge of the conceptual and procedural knowledge that students might bring to a particular learning situation. She shows an awareness of the stages of understanding through which the students might proceed to develop their conceptual thinking. She selects useful and powerful problems, illustrations, and representations to help students learn new concepts. Shulman believed that teachers need to know "ways of representing and formulating the subject that make it comprehensible to others. . . . [They need to know] alternate forms of representation . . . [and] an understanding of what makes the learning of specific topics easier or difficult" (p. 9).

Mrs. Clark shows her awareness of the difficulties that students might encounter with new learning. However, she is not always sure where the students might go in the discussions. She says that setting up these types of problem-solving situations can be risky. She says, "[However], it is okay [for me] to take a risk in front of these children. I am asking them to take big risks by keeping on delving in their thinking with me." When students do encounter difficulties, she thinks of and uses strategies to help students overcome the difficulties and help them connect their old thinking with new knowledge.

Mrs. Clark's knowledge of the subject matter and how children learn the content of mathematics is central to her teaching. Her subject matter knowledge is strong enough that she is not threatened by the students' questions nor afraid to admit that she does not know. She is comfortable in taking risks that allow students to make and test conjectures that may be beyond ones about which she has thought. Mrs. Clark uses the knowledge of her children's mathematical thinking to help them learn about their own thinking. With this knowledge she helps them learn to clarify their thinking. In her instruction she pays attention to individual students' thinking while helping all the students to increase their mathematical power. Mrs. Clark says,

> I think a teacher needs to understand how children learn--that they are not going to be exactly the same. You have got to vary the way that you teach so that you include everyone. You have got to provide experiences that meet all the needs of your children.

Fennema and Carpenter (1992) emphasized that teachers must have knowledge of how children learn mathematics. They write that teachers "must have knowledge of problem difficulty as well as knowledge of distinctions between problems that result in different processes of solutions" (p. 2). These researchers noted that teachers must help students relate new knowledge to the informal knowledge that they already possess.

In most of my discussions with Mrs. Clark, she talked about her students' learning. She was very concerned that they learn to think for themselves, be able to articulate their thinking, and build confidence in their own thinking. She talked about particular children and how they had improved in thinking and in communicating their thinking. She talked about her concerns in trying to help particular students in building their mathematical knowledge and become more engaged in their own learning. Her descriptions of particular students were rich. For example, in the following interview Mrs. Clark demonstrates a clear image of one student's thinking.

> I think that Ann has been showing a steady improvement. It has been slower than some [other students]. She lacked confidence in herself and her thinking. Ann went through having no confidence to answering things a little bit. I think that she thought of the answers, but would not take the risk to raise her hand Then when someone else would say her answer, she would realize, "Somebody else explained what I was thinking, but I didn't know how to say it. And it was right." She began to gain confidence. Then she tried answering a few times. Now, I think she has gotten to the point that she just answers. With the confidence that she has gotten, she has really looked at things harder.

Mrs. Clark places a high priority on choosing the appropriate instruction to help individual students gain their potential level of mathematics power. However, despite her understanding of each student's differences in thinking, Mrs. Clark was surprised and often astonished by her students' capacity to think at high levels. She said that the most important thing that teachers should remember is "Don't underestimate children's ability to think." She recognized that her students could learn about and do many things that she had never expected of them.

Mrs. Clark is very knowledgeable about curricular mathematics materials--the range of instructional materials that are available to teach mathematics. This knowledge is evident within the classroom. She makes use of many books, manipulatives, and alternative materials. For instance, she uses grid paper to help improve students' concepts about perimeter and area, and multiplication and division. She chooses manipulatives such as solid figures and pattern blocks to increase spatial thinking abilities. She uses technology such as calculators and computers to enhance her teaching. Her students use calculators as thinking tools everyday during the class.

Mrs. Clark's Teaching Dilemmas

According to Clark (1988), teaching is "complex, uncertain, and peppered with dilemmas" (p. 9). He writes that much of teachers' thinking "energy goes into trying to predict and anticipate potential problems, to guess and estimate what students already know and how they might respond, and to forming plans and routines" (p. 9). Lampert (1985) defined a dilemma simply "as an argument with oneself" (p. 182). However, together Clark and Lampert (1986) gave a more comprehensive definition of dilemmas of teaching, "the teacher encounters a host of interrelated and competing decision situations both while planning and during teaching. There are no perfect or optimal solutions to these decisions" (p. 28).

Mrs. Clark openly discussed the dilemmas she has faced while seeking to obtain a balance between depth and breadth of the mathematics content. She recognizes that she is sometimes limited in the quality of experiences she can provide for her students. Most of her concerns are about the students themselves. Are they learning correct knowledge? Am I meeting each student's needs? She said that each class is different and that she must change her approach somewhat every year to adapt to the class. One of Mrs. Clark's primary dilemmas is how to get the students to move from old ways of thinking about mathematics to new ways. How can she help them to begin to communicate their thinking?

She talked about how the students were slow to communicate their thinking at the beginning of the year.

> You have to respond to each class. At the beginning of the year the children were not used to interacting during the math class. They wanted to give cut and dry answers. They wanted it to be the end and go on.

Mrs. Clark thinks that after several years in classrooms in which teachers taught mathematics through drill and practice of computations, students are very slow to take risks to explain their thinking about solutions. She struggles with assuring students that she will accept their answers and their ideas without making judgments.

Mrs. Clark also discussed the dilemma of not having enough resources available to teach mathematics in ways that she knew would improve upon students' opportunities to construct correct knowledge. She did not see any of these resource constraints as insurmountable and had learned to teach within them. She said that she had to spend a lot of time in the first few years making and collecting materials on her own. She says, "I have to buy things on my own or come up with some materials on my own. I collect things, like egg cartons. I've slowly built a 'manipulatives pile.' In terms of me, this is not such a problem. But in terms of other teachers--some are not so willing to do anything about [not having resources]."

One dilemma with which Mrs. Clark has learned to cope concerns the uncertainties that come when she allows students' thinking to help direct the lessons. She said that a teacher needs to be comfortable taking risks in front of the teachers. Connected to this dilemma is another aspect where teachers need to become comfortable when they teach with constructivist approaches. When students are constructing knowledge through their interactions in the classroom, there is more "noise." Mrs. Clark says,

> Many people think of [doing] math as a quiet thing. Teaching this way (students interacting and collaborating together), you have more noise level. You are going to have some off-task behavior that you are going to deal with. It's easier for many teachers to say, "Listen to me explain it. Take out your book and a piece of paper." Then have them do the assignment. Walk around the room and help. All this is

> quieter and easier than having to deal with some off-task behavior, like doing things with the manipulatives that you do not want them to do.

Perhaps, the most serious dilemma that Mrs. Clark reports concerns assessment--how to understand and interpret her students' thinking. Sometimes the students are not clear when they explain their thinking. She thinks that it has been difficult to learn how to understand what they say. She says, "It's hard to respond to the students' responses--understanding and trying to work with [what they say]. I had a hard time with understanding in the beginning." Since sometimes it is difficult to understand what they say, Mrs. Clark often needs to help students find other ways to explain their thinking (i.e., drawing diagrams, using manipulatives). Because she wants to build her instruction on students' prior knowledge, resolving this dilemma is extremely important. Mrs. Clark also says she sometimes has difficulty in knowing how far she can help students to extend their thinking. Here again she discusses how important it is to know your students. She says,

> How far can you take one child [in extending his/her thinking]. Some of it depends on the child. You have to know the children. [For instance], sometimes with Emmy, I know I've taken her so far and then I have lost her. So I drop it. Whereas with Hal, he might keep challenging me more and really want to know more.

Teaching in ways that not only help children construct mathematics but also to construct *correct* mathematical knowledge clearly intensifies the dilemmas that must be resolved. Lampert maintained that accepting that there will be dilemmas and coping with them is better than trying to resolve every one. Learning to manage gives teachers power to shape the direction and the outcomes of their work. "Facing a dilemma need not result in a forced choice" (Lampert, 1985, p. 182). Mrs. Clark manages her dilemmas concerning teaching mathematics by remembering what she thinks her role as teacher should be--helping her students learn to think mathematically and value their own thinking.

> My role is to insure that what they are seeing in whatever we are doing, whether it is hands-on or going out looking at things or the geoboards, is brought to some meaning and some understanding that can be applied in real life situations. I [try] to help them clarify their ideas. Sometimes I don't do it quite as well as others because I get confused with what they are saying, but I'm interested in clarifying

> their ideas and sometimes, if they don't agree that I've done it correctly they will respond back.

Mrs. Clark wants her students to have quality learning experiences. She says that she sometimes gets her class started in a particular direction because she wants to encourage them in specific problem-solving strategies or guide their thinking into certain new ideas. Other times she says she simply gives them a problem to solve and waits to see where the students take their learning. Mrs. Clark says, "The key to getting children to improve their problem solving ability is first to make sure teachers are committed to the role of helping students understand."

Planning for Instruction

According to Clark (1988), to understand teacher planning is to understand how teachers transform and interpret their knowledge as they prepare for instruction. To study teacher planning can help explain why and how the teacher makes curriculum and instructional choices for classroom instruction. Researchers on teacher thinking and decision making have attempted to differentiate and separate teachers' preactive, interactive, and postactive phases of teacher's planning (Brown & Borko, 1992; Clark, 1988; Jackson, 1968). However, Shavelson and Stern (1981) suggested that there is no differentiation in a teacher's thinking before, during, and after instruction. These researchers wrote that the preactive, interactive, and postactive decision-making processes of teachers are related components in which teachers develop and enact agendas. I choose to agree with Shavelson and Stern; therefore, I discuss Mrs. Clark's thinking by not separating her thinking processes into categories or phases of planning.

Mrs. Clark's knowledge and beliefs about mathematics play out in her thinking about instruction and classroom practice. She constantly analyzes and examines her teaching. She makes planning decisions before class based on the information that she has collected about what each child knows and how she can structure her teaching to provide the means for each child to learn.

Role of Plans

Mrs. Clark uses her plans to get her students going, but she is very flexible in the direction that the lessons take and does not get upset if they do not talk about or investigate some areas that she might have intended for the lesson. She said, "I just set it up and see what happens." The only result that upsets her about the direction of a lesson is if no real content has been discussed. She said, "When I plan a lesson like the day we went outside to look at parallel lines, I don't want to just stop there. Big deal--we found some parallel lines. I want them to go further with the content." That day, after the students came back to math class, she involved them in a discussion about how a rhombus and a rectangle (both having parallel lines) are different.

Mrs. Clark uses her plans in order to prepare herself well. She says that sometimes she becomes so involved during teaching the lesson that she forgets some of the thoughts that she had while planning. So sometimes she makes notecards with questions that she wants to remember to ask to cause students to think. She explains the notecards in this way,

> Sometimes I do go into a lesson with very specific thoughts. Sometimes I have notecards that might contain something that I thought would be a neat thing to do. I get going and sometimes forget. So the notecards help me with what I want to do for the concept development. Also, it's funny, when I write them down I usually can remember them.

Even though Mrs. Clark does not have a planned script for lessons, she plans her lessons to begin with a learning situation that involves completing a task or solving a problem. Even when she makes decisions about her instruction beforehand, she is responsive to the students' thinking and ideas. Mrs. Clark searches for ways to see what the students already know. The students' answers help her know where to guide the lesson. However, as Mrs. Clark explains, the process is not simple. She says,

> Sometimes you wait and see and sometimes you make the decisions ahead of time. Sometimes I know what I want the end result to be and I know where I am going to begin, but I really just want to kind of let what happens happen in between. I don't

> have a clear cut sequence of events. I usually feel comfortable with where the students are and how it is going to happen.

She is willing to change instruction from her previous decisions. Sometimes the lesson involves more thinking and interaction than she expected. Often the students surprise her and do not answer a question the way she thinks they might. An example of when she allowed students to change the direction of the lesson was one morning during May. Mrs. Clark had written on the chalkboard the following problem:

> Suppose you are entering a contest for a bicycle.
>
> 1. Does the number of people who enter affect your chances of winning?
> 2. Why or why not?
> 3. Give examples of contests, drawings, etc., that you can be a part of with only a chance of winning.

Missy, a student in the class, said that she thought that it would depend on how fast you could ride your bicycle. Obviously, she thought that the contest was a bicycle race. Mrs. Clark took advantage of Missy's answer to channel the discussion into very "rich" conversation about probability. She changed the problem to--what are your chances of winning a bicycle race? During the discussion which lasted about 20 minutes, the students discussed several important variables that would be involved in considering this problem. Some of those variables were the number of people entering the race, your mental thoughts before and during the race, the length of the race, and how well you could ride your bicycle. She later brought them back to the direction that she had intended to go with the question--a contest involving a drawing for a bicycle. However, their previous discussion had added depth to the lesson by providing a new way to look at the problem.

When Mrs. Clark makes planning decisions she considers several factors. Some factors that she considers are (a) the potential difficulties students might encounter with specific concepts, (b) the learning needs and interests of students, (c) the concepts that she wants students to learn, (d) the types of problems or the learning situations that will help teach the concepts, (e) the difficulties of problems, (f) her knowledge of students' prior knowledge, (g) the strategies she will use to present problems, (h) the assessment of students, (i) the materials that students will use to solve problems, (j) the time allowances for concepts or topics, and (k) the organization of students. In the following sections, I will discuss several of the previous factors that Mrs. Clark considers when formulating her plans.

Potential Difficulties

One factor Mrs. Clark considers is the potential difficulty that students might encounter in learning specific concepts. For instance, in a lesson on spatial transformation that I will later discuss, she planned for the students to use grid paper so that they could see more easily the transformations that they performed with objects (pattern blocks). If students' understanding began to falter, Mrs. Clark had planned ways that she could adjust her instruction during the experiences in the classroom.

Another example of when she planned ahead to provide help if students encountered difficulties was when she was teaching about the concept of fraction. She said that students often have difficulty in the beginning with this concept. Students had opportunities to discuss fractions all year long in one way or another, but on this particular day, Mrs. Clark had planned to *officially* begin a unit on fractions. She had planned this day's lesson around the question, "What does the fraction 1/2 mean to you?" She thought that students would give the answer, "one of two parts." However, when she actually began teaching the lesson, things went differently. She asked the question, "What does the fraction 1/2 mean to you?" Students gave several different answers--half of one, half of a whole, in the middle of, two parts that have the same amount, two parts that equal up to a whole. No one said the expected answer, "one of two parts." She finally said, "You are too smart. I thought you would say one of two parts." She brought out a piece of gum and split it into two parts that were not equal. She said, "I was going to keep the bigger piece and give one of you the smaller piece. Then, I was going to say, 'That's one of two parts. Is that 1/2?' But you guys are so smart, I can skip all that." (She had planned several other items to facilitate their thinking about 1/2). She later said of this lesson,

> I built my lesson around [the initial question of] "What does 1/2 mean to you?" I expected them to say something and they didn't. Although I anticipated them saying that (one of two parts), and I was prepared for that with a piece of gum--the little piece and the big piece and other things. It still worked out. I didn't plan it so strictly that I said, "Now, what will I do?" So I said, "Boy, you guys are so smart. You have already got it. I thought you were going to do this." That makes them feel good and they can say, "You didn't trick us."

Needs and Interests of Students

When Mrs. Clark plans her lessons she also considers the learning needs and interests of the students. She says, "I am familiar with what fourth graders should do. I ask, 'Is it meeting individual needs?' I also think of what is going to be meaningful, what's interesting, and what's purposeful to the children."

Types of Problems

The most significant factor in her planning for instruction is the type of problems that she chooses. Mrs. Clark chooses problems that challenge the students to think. Her choice of problems begins the students' thinking and provides the context for the students to construct knowledge about the content that she wants them to learn. Because of her planned problem choices and tasks in which to involve students, she hopes to guide the students in not only constructing knowledge but also in constructing *correct* knowledge.

Time Decisions

Mrs. Clark is very flexible about how long it might take to talk about one problem. Teachers who teach with a constructivist approach give students time to think and talk about problems (Brooks & Brooks, 1993). She is also flexible about the amount of time she spends teaching mathematics; however, mathematics class never lasted less than an hour. Mrs. Clark decided to teach mathematics at the first period of the day when the students were rested and refreshed. Mathematics class *officially* begins at 8:00. Students arrive between 7:40 and 7:45. They begin writing in their journals and working on solutions to the problems on the chalkboard. At 8:00, Mrs. Clark begins class with students discussing their solutions. Most of the time mathematics class is over by 9:10. If the interaction and discussions become very productive, however, she might decide to continue until 9:30. So, ordinarily, math class lasts about 70 minutes. Mrs. Clark talks about how she makes the decision to extend or end class. "Sometimes I drop things, but sometimes you can't just drop things. It puts you in a real decision making role, as far as how far you can continue something."

If they don't have time to do something in class that Mrs. Clark really thinks is important, she will try to make up for it later in the day. Otherwise, the next day she will pick up where she left off. She says, "It is interesting. Sometimes I get carried away when I am teaching. I have so much that I want them to think about. I think that we can get it all done. But we will get so involved. However, I can usually figure out what to do during the day to make up those minutes."

Decisions for Organizing Students

Mrs. Clark uses the nature of the activity to help her decide how she will organize the students. Mrs. Clark believes that children of all ability levels should work together. She believes that low-achieving and high-achieving students should work with all the other students. She believes that her students learn more if they stay with their classmates. All students in her class learn mathematics at the same time working on the same tasks or activities. The NCTM (1989, 1991) recommended that students in the same grade learn mathematics all together rather than in ability groups and that the same topics and activities be used. Different types of classroom organization that Mrs. Clark uses are cooperative groups, individuals working independently, and whole group.

Cooperative groups. Sometimes the cooperative group is four or five students or a pair of students. For instance, one day Mrs. Clark paired a student who understood the division algorithm process with a weaker student. She said, “When I put together a group, it is not necessarily higher achieving students and lower achieving students but stronger thinkers and weaker thinkers.” She says that this arrangement benefits both students. The stronger student will “clarify his or her thinking while explaining the process of dividing, multiplying, subtracting, or whatever. The weaker student gets the extra experience of practicing the algorithm.” Cooperative groups make it possible for different perspectives to be heard (Yackel, Cobb, & Wood, 1991). Sometimes Mrs. Clark puts weaker thinkers together so that they are “forced to communicate to solve the problems.” She says, “They have to say the words to explain it.” She sometimes puts more vocal students together because they are “forced to listen to other people’s ideas.”

Mrs. Clark thinks that cooperative groups help students learn to work together. According to Johnson and Johnson (1989), students learn to use interpersonal skills that will help them learn to cooperate to solve problems. Mrs. Clark moves around listening to how the students are cooperating when they are working in cooperative groups. She uses these observations to make decisions about who will work together in future activities. To help students learn to cooperate to find an answer, she often refers to a solution as the group’s solution when she brings the class back to whole-group discussions at the end of activities. She wants students to work together to agree on their approaches and answers.

Mrs. Clark wants the students to explore and listen to each other’s thinking. She often decides to group the students together when they are working on word problems. She says that students encourage each other when they are solving problems in cooperative groups. She says, “If they are strong in understanding something, I think it is good for them to help others to understand.” For example, when they were studying measurement, the students worked in cooperative groups because she wanted them to discuss estimation and investigate together how to use the different measuring units--standard or nonstandard. They learned these skills and ways of thinking through discussing and negotiating ways of measuring and estimating with other members of their group. Johnson and Johnson (1989) noted that "if mathematics instruction is to help students think mathematically, understand the connections among various mathematical facts and procedures, and be able to apply formal mathematical knowledge meaningfully, cooperative learning must be used in mathematics classes" (p. 235). These researchers also say that cooperative learning is essential if teachers are to encourage students in problem solving, in communicating and reasoning, and in developing self confidence.

Mrs. Clark likes to use cooperative groups so that she can listen to the students as they talk to each other. She says, "I like the interaction within [the groups]." However, her belief that children learn to think mathematically while they interact with each other guides her use of cooperative grouping. She says, "A lot of people think of math as a quiet thing. You have [a higher] noise level with cooperative groups."

Working independently. In order for Mrs. Clark to observe what each student is understanding mathematically, she sometimes has them work independently. When Mrs. Clark wants to observe her students individually, she often places them in a horseshoe shape. For example, when she began a unit on fractions, she moved the students into a horseshoe shape where she could observe the students working with individual manipulatives. She said, "It is important to see what each child is doing with their individual manipulative." When students write in their math logs, she asks them to work individually because she wants to observe and evaluate what they know without the help of others. She also wants the students to focus on what they already know about the problems. She wants them to try to develop their own thinking about the solutions.

Whole group. At least half of the time, Mrs. Clark uses whole-group instruction to teach mathematics. She brings students back to whole-group instruction from cooperative groups when she wants each group to present their strategies and solutions for solving problems. She believes that the students must talk about what they have learned in cooperative groups. She wants them to report on questions of interest that they have encountered while working cooperatively. When Mrs. Clark leads the discussion on the calendar, the daily deposit, and the counting tape (discussed in Chapter IV), she always uses whole-group instruction. She wants everyone to have the opportunity to hear all the explanations. Yackel, Cobb, and Wood (1991) stated that during whole-group instruction a teacher can encourage in-depth discussions among children as they explain their mathematical thinking. These discussions are important for students' mathematical development as they socially interact. These researchers added that whole-group discussion after students have worked in cooperative groups is valuable for students to explain their solution attempts and get other students' interpretations on those attempts.

Mrs. Clark also organizes the students in whole group to give them opportunities to discuss ideas that they have developed individually. She believes that the students learn to construct correct knowledge when they justify their thinking. She thinks that the students clarify their thinking and deepen their understanding by making connections from their own thinking to that of others. Mrs. Clark wants them to communicate in whole groups in order to gain confidence in their ideas. She thinks that when students learn to speak about their learning, they take responsibility for their thinking rather than depending on the teacher.

Content Decisions

Mrs. Clark's plans include an agenda about the content or concepts the students will be discussing. Mrs. Clark's planning about which concepts or topics to teach is guided by what she thinks that fourth graders should know, what they need to know for the fifth grade, and what they find interesting. She also uses *The Standards* (NCTM, 1989, 1991) to help her interpret the focus of content.

Central to Mrs. Clark's teaching is her ability to decide on mathematical tasks or problems that engage the students in creating and using their mathematical knowledge. She poses real-life, interesting, and challenging problems for the students to solve. Lampert

(1989) noted that research on mathematics learning suggests that in order for teachers to create instructional situations in which students construct *meaning* to mathematical procedures, learners must engage in real-world problems with a solutions they care about.

Lappan (1993) explained the importance of worthwhile mathematical tasks in the following quote:

> No other decision that teachers make has a greater impact on students' opportunity to learn and on their perceptions about what mathematics is than the selection or creation of the tasks with which the teacher engages the students in studying mathematics. (p. 524)

Teachers need to pose problems that are relevant for their students. These problems are a guiding principle of constructivist teaching and learning (Yackel, Cobb, Wood, Wheatley, & Merkel, 1990). The problem-solving tasks that Mrs. Clark chooses help the students learn mathematical thinking and reasoning. Mrs. Clark says, "Situations involving problem solving should occur daily and should relate directly to what children experience."

Tasks to invent algorithms. Mrs. Clark chooses problem that provide opportunities for students to develop relational understanding about mathematics. According to Hiebert and Wearne (1986), when students learn to relate the procedures to the concepts, then their thinking becomes more powerful. When Mrs. Clark involves students in problems or problem-solving situations where they learn procedures for solving problems or learning algorithms, she chooses mathematical tasks in which students can invent these procedures on their own or with her.

Mrs. Clark chooses problems and instructional activities so that students can make generalizations about rules for symbols or algorithms for themselves. During my observations, they developed algorithms for finding equivalent fractions and ordering fractions and constructed algebraic equations. In Chapter V, I discussed how students invented an algorithm for changing mixed numbers to improper fractions and for ratios and proportions in probability. These tasks help students construct meaning and build relationships between conceptual and procedural knowledge. Skemp (1978) stated that connections between procedural and conceptual knowledge create a relational understanding of mathematics.

Mrs. Clark wants the problems to create situations in which it is possible for the whole class to carry on discussions that will produce a variety of approaches for developing procedures. She plans tasks that will stimulate the students' creativity and emphasize real learning rather than simply applying some algorithm.

Tasks that challenge students. Even after problems are solved, students sometimes continue to investigate and think about the problems. Throughout the day students will talk among themselves and with the teacher about their thinking. Students and groups of students in class or outside of class become genuinely absorbed by the problems. Mrs. Clark says that she likes to choose problems to which the students will respond well and enjoy. She likes to choose a variety of problems. Most importantly, however, she chooses problems that will challenge the students. In the following excerpt from an interview, Mrs. Clark talks about why she chooses certain problems.

> I want it [the problem] to be challenging to them. I don't want them to get the answer really quick. They like it when it's challenging. When they [the problems] are easy, they lose interest because they are not really thinking. They say, "Well, everybody is going to get that answer." They think, with the more challenging things, that their answers [will be] unique. They can still have a different answer than someone else. I choose activities where they will respond well.

Yackel, Cobb, and Wood (1991) noted that teachers should design instructional activities that are likely to be problematic for children. Teachers who base their instruction on children's learning of the concepts facilitate disequilibrium, cognitive conflict with present knowledge, and problem solving. They added that "a personally challenging problem is more important than completing a large number of activities" (p. 197).

Tasks that develop strategies. Mrs. Clark provides her students with experiences to learn different problem-solving strategies. She says,

> Children should be continually made aware of problem-solving strategies, such as drawing a picture, guess and check, and working backwards. They should be allowed to choose how they want to go about coming up with an answer and use that to defend their reasoning.

Mrs. Clark believes that if students select and use appropriate problem-solving strategies they will learn to reason mathematically. By analyzing the problems and choosing problem-solving strategies, students make sense of mathematics. "Listening to their peers and their teacher describe other strategies helps students refine their thoughts and . . . express their thoughts" (NCTM, 1989, p. 82). Problems that she chooses help encourage students to create their own questions about the problems, provide proof to support their explanations, compare their explanations with other students, and make conclusions about the quality of their explanations or solutions. Mrs. Clark chooses problems that will encourage the students to investigate several solutions. She thinks that the process of investigating problems is more important that the end product or answer. Mrs. Clark poses problems that are complex enough to draw out multiple solution strategies.

Tasks that facilitate assessment. Mrs. Clark chooses problems that require students to think in enough depth that she can observe their reasoning strategies through their verbalization of their thinking, their use of manipulative materials, their use of paper-pencil diagrams and pictures, or their writing in their math logs. She said, "Students [should be] heavily involved in good, solid problem solving experiences. They should be encouraged to communicate their thinking both orally and in writing." The tasks that she chooses help her understand both the students' prior knowledge and provide for future instruction. She decides if the tasks are effective in helping students learn mathematics. Mrs. Clark looks for tasks that stimulate deep discussions that help her assess the mathematical power of the students.

Webb and Romberg (1992) suggested that since mathematical knowledge is dynamic and complex in nature, the students should be involved in tasks or problems where they investigate, explore, reason, and make connections. These problems help students learn that mathematics is an integrated whole. A teacher who uses these types of tasks for students to solve will be able to assess students' understanding of relationships among mathematical properties.

Real-life problems. Mrs. Clark often chooses problems that are set within the context of a familiar real situation. She chooses real-world situations or problems for students to solve so that the students can make sense of the problems. The problems allow students to move into unfamiliar territory while at the same time using familiar concepts or ideas that help them to verify whether they are proceeding in the correct direction. Mrs. Clark extends the familiar content by broadening and relating the concepts the students already know to more difficult content. As students talk about their strategies and reasoning for finding answers, Mrs. Clark introduces topics that parallel this familiar conceptual territory. Some problems that she has chosen for the students to investigate to achieve these goals are as follows:

1. If rosebushes were $4.59 each and there are 3 for $13.50. Which is a better buy?

From this problem, several issues came up during the discussion. They explored multiplication and division. They discussed several important variables--whether the plants were fresh or not and the number of plants that a person might need.

2. Mrs. Boyer wants to plant rosebushes along both sides of an 8-foot path. If she plants them 1 foot apart, how many rosebushes does she need?

Students learned to consider the importance of symmetry in landscaping. From this point, they decided that a rosebush should be placed at the beginning and the end of the walk. Several students drew diagrams of how the path would look. One student showed his drawing--one that was drawn to the scale of 1 inch equal 1 foot. Mrs. Clark took this occasion to show them how to make scale drawings.

> 3. At Spencer's Diner you get a free lunch after every 8 lunches you buy. If you ate there 45 times last year, how many of your lunches were free?

This was another occasion when students solved the problem in different ways. They discussed the importance of keeping their thinking clear by showing some kind of order.

8 paid dinners--1 free

8 paid dinners--1 free

8 paid dinners--1 free

8 paid dinners--1 free

8 paid dinners--1 free

Students used all the operations to solve the problem. One student solved the problem by subtracting 8 from 45 successively. Several students used addition. Some students used multiplication or division. Mrs. Clark emphasized that there was not one correct way.

> 4. Store A sells 15 cards for $12.00 and Store B sells 10 cards for $8.00. Store A is 10 miles away and Store B is 20 minutes away. Which would you buy the cards in and why?

Even though some students used their calculators on this one, none of them figured that Store A's cards were 80¢ per card by dividing--$12.00/15. They found the answer to the problem in an unusual way. It was interesting how the conversation led to questions that showed that they were thinking in terms of the concept of rate. They could not figure out how long it would take to go 10 miles (distance of Store A) though they were sure it must take more than 20 minutes (time to get to Store B). Some of the students started saying, “Well, it depends on how fast you can go.” Then to find out about the rate, they asked questions of Mrs. Clark like, “How far do you live away?” and “How long does it take you to get here?”

Sources of Problems

Mrs. Clark sometimes chooses the content of her lessons through reactions to a student’s statement or question. For example, one student had stated that he did not find what they were doing with measurement to be very useful--changing inches to feet to yards. She used this doubt to plan the next day’s lesson. During the next day’s activity she asked 11 students to come up front and hold out their rulers until they reached 132 inches. (They were on day 11 of the month. They had been adding a foot for every day.) She then laid three yard sticks on top of the rulers and asked, ”What do you think about that?” They decided during this discussion that there were 2 feet left over without a yardstick laying on top of them. That meant there were 3 2/3 yards. From this demonstration, they decided that 132 inches = 11 feet = 3 2/3 yards. She concluded the discussion by asking the students to brainstorm ways that knowing how to change inches to feet to yards could be helpful. She talked about this in the following statement.

> Yesterday, when Frankie indicated that he didn’t think that having to change from inches to feet or to yards would be useful to know, I decided to have them look at that in today’s measurement activity. I knew that we would be looking at 132 inches. I planned this problem because it was divisible by 12 and also divisible by 3 (12 inches in a foot, 3 feet in a yard). When I saw that I was dividing by 3, it immediately clicked because they know the divisibility rule for 3. So we could look at yards.

When Mrs. Clark plans her lessons, she does not use just the teacher’s manual. Even though she looks at the chapter to see what the book is emphasizing she uses a variety of sources to develop her plans. She does not proceed directly through the book. She does have a scope of concepts and skills that she wants the students to learn, but she does not have a planned sequence. She orders the concepts that she will teach based upon the concepts the children already understand. Duckworth (1987) suggested that teachers who teach with the constructivist approach to learning do not often prescribe a scope, sequence, or a timeline for learning topics. All these can interfere with a student's ability to understand complicated concepts.

Mrs. Clark sometimes introduces the students to new ideas and concepts that they will be learning in more depth later in the year. However she chooses to sequence the content, by the end of the year, she has taught every concept or procedure that is in the textbook. She does use some of the enrichment activities in the teacher's manual. She also assigns homework from the book. When I asked her where she gets the ideas that she uses to plan her instruction, she responded,

> I just read and build up a storage system in my mind. For instance, if I am doing angles, I think, "Now what of all of this bank of knowledge that I have about angles do I think is going to work best in getting these children to learn." I will think of a neat idea and say, "That will be a neat thing to do." It may be something that I have read or something that just comes to me. Lots of times I go to bed at night and I will start thinking.

Mrs. Clark subscribes to *Arithmetic Teacher*. She has a bookshelf full of books with ideas on activities for learning mathematics. She has several books with interesting problems for challenging students to think. She specifically recommends McDougal and Littel's *Daily Mathematics* Series. She has developed with another teacher mathematics problems and activities that she calls *100 Days of Math.*

> I get problems from a variety sources. I use the *Problem of the Day* booklet and chart that came with the textbook. Last year I wrote a *100 Days of Math* program with another teacher at school. I have a variety of books that I use. And ideas come to mind sometimes that are related to what we are studying in other subjects. Sometimes questions students ask will bring a problem idea to mind. Or the students also like to contribute their own ideas. This is encouraged by me and makes them feel really special and proud.

Mrs. Clark often uses her own life experiences as a resource in planning problem-solving situations for the students to investigate. Mrs. Clark sees mathematics everywhere, especially in everyday situations. For example, she made up a problem from her experience one weekend. She and her family had taken their RV on a camping trip. On their way they had stopped at an RV supply store to get some items. When she was ready to check out, a man in front of her was completing his purchase. He said to the clerk, "Why, you gave me back the same number of bills that I gave you." His purchase had been $282.64. She said "I thought, what a great problem. I am going to tell my students the amount he purchased and see if they can up with which possible bills he gave them and they gave back to him." This was the problem that she ended up asking:

> A man purchased some material at a store. The total cost was $284.64. He gave the clerk $300.00. Upon receiving his change, he said, "I got the same number of bills back that I gave you." What bills did he give the clerk? What bills did he get back?

Decisions About Mathematical Tools

The mathematical tools in Mrs. Clark's classroom are the concrete materials or manipulatives that support the students' efforts to link the concepts with the procedures. Mrs. Clark plans activities that provide an opportunity for students to perform meaningful actions with the concrete materials. Mrs. Clark's choices of mathematical tools depend upon the concepts that she wants her students to think about. Lappan (1993) wrote that manipulative materials should offer the "potential for students to engage in sound and significant mathematics as a part of accomplishing the task" (p. 525). Manipulatives are tools to be used thoughtfully by students to help them make sense of mathematics.

Mrs. Clark values manipulative materials for students' learning. She says that to teach math so that children learn to think in ways advocated by *The Standards*, you "can't just open the book and do page such and such. [Manipulatives are] wonderful to help me generate ideas for the students to think about."

Mrs. Clark uses manipulatives such as fraction circles and fraction strips to help the students understand fractional concepts. She does not use pattern blocks to teach fractions because she thinks that they confuse the students in learning fractional concepts. She uses base-10 blocks and grid paper to help children learn about perimeter and area. She also uses base-10 blocks for addition, subtraction, multiplication and division. She uses plane and solid shapes, pattern blocks, and geoboards to help children learn to think geometrically. Mrs. Clark explains her considerations when she is choosing mathematical tools, "I know fourth-grade thinking. . . . I know what to choose to get their minds to work. I know what kinds of experiences the children need to grasp [the concept] well." Mrs. Clark chooses commercial items such as dominoes to teach mathematical concepts such as multiplication, division, and fractions. Items from real-life become one of her most important sources of learning tools. Mrs. Clark says one factor in deciding which manipulatives to use is, "I just think of what thing I can choose that would be the most effective way, the easiest way for them to understand the concept I am trying to get across."

Lampert (1989) suggested that manipulatives are important because "representations of a concept contribute to long terms understanding and knowledge use" (p. 256). Students should use manipulative materials that will help them model their mathematical thinking. Using manipulatives can help stimulate thinking and help children build their own knowledge. Manipulatives can also help to promote reflective thought (Baroody, 1989).

Using tools for assessment. Mrs. Clark says that manipulatives are not toys. They are materials that students use to help them think. Mrs. Clark requires her students to use manipulatives so that they can model their thinking. However, Carpenter, Fennema, and Franke (1992c) emphasized that valuable use of manipulatives requires that teacher or knowledgeable adult interact with the learner. Careless use of manipulatives may convince students that there are two separate mathematics, symbols and manipulative.

When the students perform actions with the manipulatives, Mrs. Clark can evaluate their understanding by observing their modeling. While the students model their thinking, she helps them clarify and justify their thinking, both to themselves and to others. She usually has enough manipulatives for each student to use. Mrs. Clark says, "When you have a manipulative for each individual, you can see what each student is doing."

The children make many of their own learning tools, such as fraction circles, fraction strips or bars, and solid figures. Mrs. Clark likes for the students to make their materials because through the process of making them she can assess what they know about the concepts. By listening to students as they make the manipulatives they will be using in future lessons, Mrs. Clark has the opportunity to find out their prior understandings. This information helps her plan the next level of the learning of each concept.

In one lesson students made a fraction circle from a paper plate for the fraction 8/8. They colored the fractional parts with different colored crayons. Later during a cooperative group lesson, students asked each other questions about their own circles. Some examples of their questions during the group activity were as follows: How many slices are blue and green? How many parts are not colored white? As students asked each other questions, Mrs. Clark walked around the room listening to the students' questions. She designed the next day's instruction using the 8/8 fraction circle based on her observations of the students' use of the manipulative.

Planning decisions when using tools. The plans that Mrs. Clark designs when students will be using manipulatives are more specific than other plans. She has a clear agenda in mind. Her plans include actions she will take to guide the students toward understanding. She is clear about what she wants the students' actions to be with the manipulatives. When Mrs. Clark thinks about what the students will be doing with manipulatives, her plans are similar to the "expert" teacher in Leinhardt's (1986, 1989) research. Leinhardt (1989) wrote that expert teachers form content-based agendas. Agendas are "unique operational plans that a teacher uses to teach a mathematics lesson.

They includes both the objectives or goals for lesson segments and the actions that can be used to achieve them" (p. 55).

In the following discussion of Mrs. Clark's plans for instruction, you will see that she has what Leinhardt talks of as a general framework of lesson segments that provide the form and structure for the teacher and the students. According to Leinhardt (1989), these segments each have unique goals, actions, and task environments. Mrs. Clark's plans for a day when she wanted the students to learn about spatial transformations through using pattern blocks demonstrate that these types of lessons are more scripted. The actions that she wanted to observe were clear. About this lesson, she says,

> I will ask them to take one of the pattern blocks and move it on their desk. While they are doing this, I will observe what kinds of things they are doing with their block as they move it. Then as I see different things being drawn, such as a slide, or a flip, of a turn, I will ask particular students who are doing one of these particular movements to come up and demonstrate what they have done on the overhead. I want them to come up with the motions and ideas of the slide, flip, and turn themselves. I want them to be able to think of the motions as things they normally do with an object. They can slide it over, they can flip it over, or they can rotate it. I hope that they will come up with a name of the movement such as slide, of flip, or turn.

Notice here that even though she has a plan about content in mind, the students' thinking exhibited through their actions upon the pattern blocks directs the lesson. Her plans included what the students would do, how they might respond, and what they might accomplish. Mrs. Clark goes on to say,

> If not, then I will work on coming up with one of those terms. After each of the motions that I am looking for has been demonstrated by the students at the overhead, I will show them how to draw that (the move from the original position to the new position) on the overhead. In particular, I hope they will see with the flip the line over which the shaped flipped, whether it was an up-down flip or a sideways flip. I will observe them making their flips, and we will discuss that in some cases it (the new position) may look different or in some cases it may look exactly the same when the motions are made. Then I will demonstrate and draw on the grid paper on the overhead how to record a slide, a flip, or a turn with a particular object. I want them to practice moving [the pattern blocks] using slides, flips, and turns on their own. The students will draw with their pattern blocks on their sheets of grid paper. They will draw each shape as they slide, flip, or turn it. The purpose of having them to record what they do is to hope they will see whether

> the shape changes or not and to see that sometimes it might look different and sometimes it might not depending on the shape they use.

The students’ actions with the pattern blocks enable them to construct their own knowledge of spatial transformations. Mrs. Clark wants her students to experience making spatial transformations, not just to visualize them from an explanation. It is performing the transformations themselves that facilitates the students’ understanding of spatial transformation.

Decisions About Assessment

Mrs. Clark believes that the purpose of assessment is to improve her instruction in order to improve students’ learning. Mrs. Clark understands that in order to develop instruction that the students need and are ready for she must listen carefully and assess the development of their thinking. While assessing her students’ thinking, she looks for what to emphasize, what the next steps should be, and how to challenge their thinking. A guiding principle of Cognitive Guided Instruction is that instructional decisions should be based on careful analyses of students’ knowledge. If a teacher structures the lessons based upon her assessment of students, the learning situation will more appropriately conform to the learner’s development (Fennema & Carpenter, 1992).

As Mrs. Clark assesses her students through listening to their explanations, she says she can understand the students’ depth of knowledge and use these explanations as beginning points to lead them to construct new understandings. Mrs. Clark believes that she can learn more about what her students understand by assessing them *while* she is teaching. She observes their interactions with her and other students and their use of manipulative materials. She says these assessments tell her more about what her students know than written assessments. She says that the complex part of understanding what students are thinking must be done by *listening* to the students' reasoning. She can assess whether this knowledge is memorized procedures or constructed knowledge that has been internalized. Lambdin (1993) made this point by noting that written tests often tell little about children's strategies. Because of this lack of information, teachers may jump to inaccurate conclusions about the children's performance. Often written tests make it difficult to diagnose children's mathematical difficulties.

Researchers, such as Huinker (1993), suggested that listening to students explanations allows teachers to gain insight into students' conceptual knowledge and reasoning during problem solving. Listening to students allows teachers to "determine the level of understanding, to diagnose misconceptions and missing connections, and to assess verbal ability to communicate mathematics knowledge" (Huinker, 1993, p. 80). With paper and pencil tasks, students' understanding is more likely to be hidden. With these tasks it is difficult to determine whether an incorrect answer is given because a student lacked the necessary knowledge or simply made an error.

Mrs. Clark thinks that this type of assessment is difficult because she is looking for conceptual knowledge which cannot be easily assessed through traditional tests. Nevertheless, she does use written assessment at times. She gives her reason for using written assessment in the following excerpt from an interview. She says,

> I think that assessing conceptual knowledge is more important than assessing the procedures, but you've got to be objective. Some assessment must be about the procedures. Students must have some sort of written feedback. Some teachers believe that you don't give any written work. They might feel comfortable with that. I am not totally comfortable with doing only informal assessment. Most of time, [however], I feel like the kinds of activities that I assess my children for grades are in line with where they are conceptually.

According to Fennema and Carpenter (1992), the more closely the form of assessment matches the level of mathematical maturation of the learner, the more useful will be the information. A key principle of assessment is to determine what mathematics the learner knows. Clarke, Stephens, and Waywood (1992) noted the most effective instructional activities are those that also provide the best assessment opportunities.

An Illustrative Example of Mrs. Clark's Thinking

Mrs. Clark's thinking about teaching is focused and coherent. Leinhardt's (1989) research supported the idea that expert teachers demonstrate with their lessons a knowledge base that is organized around centrally important ideas. In this lesson, Mrs. Clark's goals for students are that they learn to estimate, learn to measure, and learn to make sense of measurement by using it in problem-solving experiences. She has clear reasons for her choices of tasks, tools, and assessment. With the following lessons, Mrs. Clark demonstrates her thinking about the importance of conceptual knowledge. Even though Mrs. Clark demonstrates with this lesson how her teaching is focused and coherent, she also allows students the leeway to construct the direction the lesson takes when they are attempting to make sense of the problems or activities.

The illustrative example is from her unit on measurement. Notice the coherence both between each lesson and within the lessons. Mrs. Clark spent the month of February having the students learn about measuring distance, liquid capacity, and mass. When students were learning how to measure, they used various manipulative materials. She chose familiar and unfamiliar items for the students to work with to help them construct knowledge of measurement. I asked her, "What are your reasons for your choices of things to measure?" She said,

> With their first experience, I deliberately chose things that were not real (not things they would measure in real life), just experience type things. My intent was to get them to estimate and then measure. I was trying to get them to apply the use of measurement whether it was measuring distance or whether it was measuring weight. I wanted them to just practice with the materials. It didn't matter what the objects weighed. What mattered to me was that they were estimating and learning to use the measuring instrument--the tapes, or the sticks, or the ruler, or the balances, or whatever.

On the first day of the linear measurement lesson, Mrs. Clark and the students talked about all the nonstandard types of things that could be used to measure items. Some of the students suggestions included hands and sticks. They talked about the drawbacks for using these types of measuring tools. After about a 10-minute discussion, Mrs. Clark put the students into cooperative groups of five. Each group received an "interactive" box that contained one long and one short measuring unit and an index card in each box saying, "Use long and short, L or S, to measure

1. length of cardboard box
2. width of cardboard box
3. depth of cardboard box
4. circumference of lid top
5. diameter of lid top
6. radius of lid top

Mrs. Clark chose different materials for each group. In addition, each group had different measuring tools--one for the long unit and one for the short unit. Group one used the length of an empty paper towel roll for the long unit and the width (diameter) of the roll for the short. Group two used straws--a long straw for the long unit and a coffee stirrer for the short unit. Group three used egg cartons--the length of the entire carton for the long unit and the width of one egg cup for the short unit. Group four used a paper clip chain of 10 paper clips for its long measuring unit and one paper clip for the short unit. Group five used a long wooden stick for its long unit and a toothpick for the short measuring unit. Mrs. Clark said that she chose the measuring tools hoping that the students would make associations between the long and the short units, such as one paper clip is 1/10 of the paper clip chain or the long is 10 shorts.

Before they began measuring the items with their tools, Mrs. Clark asked them to estimate the measurements of their items and write the estimates on a worksheet that she had prepared. Van de Walle (1993) noted that estimation should never be taught as a separate topic. He wrote that teachers should include estimation in some form in every activity that they do. Mrs. Clark believed that estimating was the most important thing that students could do to develop a sense about measurement. Through estimation, she wanted students to develop benchmarks for the comparison of inches and feet and yards and the comparison of these to centimeters, decimeters, and meters. By requiring students to estimate before they measured, the measurement activities were not only about learning to use the tools but also learning to reflect about the meaning of what they were doing. She believed that this reflection helped them develop conceptual knowledge of the procedures. During the activity the students talked animatedly with each other arguing about the estimates and later about the actual measurements. They argued about the meanings of circumference, diameter, and radius. If some groups finished early with the items in their interactive boxes, they could measure other items in the room, such as the chalkboard and door frame.

After all the groups had finished with their activities in the interactive boxes, Mrs. Clark guided a whole-group discussion where students talked about points such as these. What about your L and S units made them easy to use? What about L and S made them hard to use? Will your measurements be the same as the other groups? How did you feel about your estimating at the beginning? Is it okay to be wrong when you are estimating?

During the discussion the group of students with the egg cartons decided that their units were hard to use because they could not easily find parts (halves and fourths) of the small unit--the egg cup. The group with the sticks thought their units were not flexible enough. The group with the straws did not like having to string their straws together to measure long items, like chalkboards. They also kept losing their place because the string kept moving. The group with the long chain of paper clips said their chain kept moving also. They thought they were *too* flexible. I asked Mrs. Clark why she chose the measuring units that she had, she said,

> I thought it would be more fun if they used different things. I wanted them to measure with non-standard units that were both "good" things and "bad." I wanted to see with the toilet paper and paper towel rolls, since they didn't have a real short unit, if that made it hard to measure small distances. Whereas, the groups that had the paper clips didn't have as much trouble measuring a shorter distance because it was a smaller unit. Also, some of the objects that I chose were more flexible and some were not. Some were harder to visualize what would be a half, a

> fourth. The egg carton would have been easily divided into sixths. The thing that I was trying to get them to understand was that there were ways in which their units were good to use, easy to use, or hard to use.

The next day in cooperative groups the students measured with standard measuring tools objects they would use in real life. Each group in their "interactive" box had a tape measure, a 6-centimeter ruler, a 12-inch ruler, a string, and a calculator. One person from each group went to get a yardstick and a meter stick. Again, all groups estimated before they actually measured. They measured two dog leashes-one thin and long, one thick and short. A card in their interactive boxes also asked them to decide which would be for a big dog? Which for a small dog? They measured 8 mm movie projector reels--one large and one small. A question on the card asked if certain size movies in canisters would fit on their reels. They measured bath towels to decide how much trim would be needed to decorate the towels. They measured baking pans. Finally, they measured an art print to decide what size frame they would need. Notice that even when students are being asked to measure, Mrs. Clark also asks them to think about uses (i.e., Which leash is for a small dog?). When I asked Mrs. Clark about her choice of items this time, she said,

> When we measured real objects, I wanted them to know that we do measure real things. We do buy things by weight. We do buy things according to the size, the length, and how much we need of those materials. If you needed an 8 X 10 frame, what did that mean? If you were putting stamps on an envelope, why would you put 1 or 3 or 4? Was it because the mass of the envelope was heavier? (They had measured the mass of objects with balance scales in cooperative groups previously.) As far as actual objects, I was already thinking ahead to what word problems that I might want them to solve. I wanted these problems to be real-life problems. I wanted to choose things that might create a meaningful word problem or situation.

Notice here that with each activity that she plans she is already thinking ahead to what she will want students to do next. At the end of the unit, as an assessment, Mrs. Clark asked the students to solve word problems similar to their activities with the interactive boxes. In this measurement unit, she demonstrates the value of connecting mathematics to real-life applications. Mrs. Clark summarized this activity by saying, "So they estimated and measured the materials, they saw a use for them, and they solved problems where they had to apply what they had learned. That really puts it in their real-life context. Things that they'll actually experience."

Mrs. Clark believed that measurement activities should help students understand what measurement is and how to go about doing it--which tool to use, how to use the tool. She wanted them to construct a measurement sense. A measurement sense is defined as recognizing and approximating the most reasonable measuring unit to use for the activity and competence in estimating the attribute that is being measured (NCTM, 1989). The NCTM suggests that all of these capabilities should be learned by students.

As you can see, Mrs. Clark's thinking and decision making about these lessons demonstrate a focused and coherent approach in facilitating students' learning of the content. She gives them the opportunities to learn the content in their own ways, but she wants to make sure that the learning opportunities lead to construct *correct* content. Like Leinhardt's (1989) teacher, Mrs. Clark has a rich agenda that is consistent with lesson segments, yet is flexible. Mrs. Clark's plans are to meet her goals of helping students clarify the concepts and procedures and having student learn and understand. Leinhardt says that "expert mathematics teachers weave a series of lessons together to form an instructional topic in ways that consistently build upon and advance material introduced in prior lessons" (p. 73).

Summary

In this chapter, I have presented Mrs. Clark's goals for students in learning mathematics. I have described her thinking about instruction and many of her instructional decisions. Through excerpts from interviews of Mrs. Clark, I have described the dilemmas that she has encountered during her teaching. A common thread--the conception of mathematics as a way of thinking--weaves through Mrs. Clark's thinking, her instructional decisions, her teaching practice, and ultimately to what the students learn about mathematics. Her knowledge of mathematics and her belief that children must construct their own knowledge is integrated within her conception of mathematics. In her mind, concepts are never isolated topics. Mrs. Clark's thinking about the relationships between the concepts and the procedures helps her students learn to apply mathematics to their everyday lives.

Mrs. Clark demonstrates with her thinking and planning that *how* mathematics should be taught is just as important as *what* is taught. With her planning and decision making, she seeks to assure that every child has success at some level in learning a relational understanding of mathematics. To show how her thinking about instruction is focused and coherent, I have detailed her reasons for choosing problems or tasks, her choices of mathematics tools, her decisions about time, her decisions about how to organize students, and her assessment decisions.

CHAPTER VII
CONCLUSIONS AND IMPLICATIONS

Conclusions

The purpose of this study was to find elementary teachers who were using constructivist learning principles in their mathematics teaching. In order to learn more about this kind of instruction, I observed and interviewed a fourth-grade teacher who not only understood how students construct their own knowledge but also knew and understood mathematics and mathematical thinking.

The outcome of this interpretative study is not to make generalizations about mathematics teaching and learning but to increase our understanding of what happens when a teacher interprets and implements constructivist learning theory in mathematics instruction. The National Council of Teachers of Mathematics developed *The Standards* based on constructivist learning theory. My study is about how one teacher, Mrs. Clark, interprets and implements *The Standards* in her classroom. This study of one teacher does not present *the* model of "constructivist teaching." In fact, the teacher herself is not aware that she is actually using constructivist learning theory. She is simply aware that she uses *The Standards* in her mathematics teaching. Mrs. Clark's approach to teaching is complex and rich and provides an example that will challenge many teachers' previous beliefs about how mathematics should be learned and taught.

In order to inquire into the teaching and learning of this fourth-grade class, I used qualitative research methods to collect and analyze data. I took field notes and audiotaped mathematics class during 4 1/2 months. I videotaped 16 hours of mathematics lessons. I conducted informal interviews and six formal interviews of the teacher. In addition, I examined teacher plan books, children's work, and other artifacts.

I have sought to answer the question, *what happens when a teacher implements constructivist teaching and learning of mathematics in an elementary classroom?* The ongoing process of data analysis using Spradley's (1980) Developmental Research Sequence revealed several conclusions:

1. A teacher's conception of mathematics influences every aspect of the teacher's teaching--decisions about instruction, plans for instruction, assessment of students, and interaction in the classroom.
2. An elementary teacher can teach using constructivist learning principles, develop a personal teaching model of strategies, and implement these strategies.
3. Using constructivist learning strategies, a teacher can create a learning environment in which students become excited about mathematics and gain confidence in their thinking.
4. By constructing their own knowledge, students gain mathematical power and become mathematical thinkers.

In previous chapters, I have described Mrs. Clarks' thinking about instruction and many of her instructional decisions. I have examined Mrs. Clark's conceptions, beliefs, and knowledge about mathematics and mathematics teaching and learning. A common thread--the conception of mathematics as a way of thinking--is woven through Mrs. Clark's thinking, her instructional decisions, her teaching practice, and ultimately to what the students learn about mathematics. Mrs. Clark's classroom goals for her students are that they learn to think mathematically and become confident in their abilities to think.

Mrs. Clark has constructed her definition of constructivist teaching and learning while creating a learning environment in which students learn correct mathematics content. Within this environment, Mrs. Clark implemented 10 teaching strategies consistent with constructivist learning theory. I used research and theory about constructivist learning to help identify these 10 strategies even though Mrs. Clark may not consciously be aware that she is using them. It may well be that another teacher would construct and use different strategies. However, Mrs. Clark's combination was very effective. Mrs. Clark structured learning by using the following 10 strategies:

1. Using concrete problem-solving contexts. Mrs. Clark involved students actively in solving open-ended nonroutine problems about primary mathematical concepts. Students learned *through* problem solving, not *for* problem solving.

2. Assessing students continually. Assessment was integrated within Mrs. Clark's teaching. Mrs. Clark primarily assessed students' conceptual knowledge with authentic meaningful activities that had relevance for them.

3. Encouraging students to communicate. Mrs. Clark urged students to verbalize their thinking about problem situations in order for them to justify and clarify their thinking.

4. Valuing students' thinking. Mrs. Clark helped students gain confidence in their thinking by affirming and validating their contributions during classroom interactions. By respecting their answers without making judgments, she encouraged students to take risks to explain their thinking.

5. Building instruction on students' articulations of their thinking. Mrs. Clark asks questions to elicit students' thinking, not to push them toward correct answers, but to probe their understanding. She listens to students' answers and builds her instruction upon their explanations.

6. Asking questions to create cognitive conflict. Mrs. Clark asks questions to help students rethink their answers or resolve their misconceptions. She acts as a moderator to help students evaluate the correctness of their answers in order for them to understand *why* their thinking is inaccurate.

7. Building on students' prior knowledge. Mrs. Clark guides students to connect their past learning to their present and future learning. She diagnoses students' levels of conceptual knowledge and helps them invent algorithms to connect to this knowledge.

8. Helping students construct a knowledge base. Mrs. Clark provides students with mathematical vocabulary and symbols to help them to think, speak, and write mathematically.

9. Encouraging reflection and helping students extend their thinking. Mrs. Clark helps students learn to analyze and push their thinking to higher levels and to new situations. When students construct new knowledge, they learn to reflect upon their old knowledge and use in new ways.

10. Teaching the students to take responsibility for their learning. Mrs. Clark guides students in proving the correctness of their own answers and does not establish herself as the only authority for correct answers. She encourages students to initiate their own learning and find their own problems without waiting for her approval.

To separate these strategies is useful for describing and analyzing how a teacher implements constructivist learning principles in her classroom; nonetheless, in order to understand the quality of Mrs. Clark's teaching, we cannot look only at the parts of her instruction. The strategies are integrated within each of Mrs. Clark's lessons. By simultaneously using the strategies, Mrs. Clark creates a rich and strong learning environment.

Mrs. Clark creates a learning environment in which the students are supported as they communicate their thinking. Mrs. Clark highlights students' thinking. She understands the types of context and activities that are crucial for productive learning. The environment is nonthreatening and supports students in discovering knowledge for themselves. Mrs. Clark verifies that each student's thinking is important and requires students to respect each other's thinking. She affirms their thinking by repeating their answers and remembering their contributions. She monitors discussions so that students can challenge each other safely. In this class, students do not laugh at other students' answers or put down other people's ideas. Mrs. Clark demonstrates that it is possible to create a noncompetitive environment in which students invent solution strategies and justify their approaches to others. The students are interested in learning and show a positive attitude toward mathematics. They become highly involved in solving problems and presenting their solutions and are excited and enthusiastic about learning mathematics. They are confident and willing to express their thinking.

Mrs. Clark helps students learn that mathematics is about investigating problem situations, making theories, and verifying these theories. Through social interaction, she provides students with opportunities to explain and describe their ideas and discover other approaches to problems besides their own. Mrs. Clark helps her students see that each person can possess a small piece of the puzzle that leads to the answer. Each person can also understand the same answer differently and at different levels. As students experience solving the problems, other problems often come out of their experiences. Mrs. Clark creates opportunities that students engage in solving genuine problems. She challenges them to think about open-ended situations that matter to them and have no easy answers. Above all, Mrs. Clark gives students opportunities to learn that mathematics is about thinking and figuring out the relationships among ideas, not about a collection of rules or procedures.

Mrs. Clark's teaching demonstrates what it means to build on student's thinking while they are engaged in *doing* mathematics. The teacher and students interact to actively construct the lessons. Mrs. Clark demonstrates that good mathematics teaching continually builds on the teacher's judgment of students' thinking. She makes decisions and solves problems concerning her teaching. She solves problems of practice much the same way that students solve mathematics problems. Teaching problems often have more than one solution. Teaching is an open-ended situation where several strategies for solving problems of practice must be used.

The problems that Mrs. Clark chooses, the questions that she asks, the choices of manipulative materials, all send messages to students about what she considers important for them to know about mathematics. Her thinking and planning for instruction is rich with complex decisions about assessment, organization of students, and time factors. The dilemmas that she discusses demonstrate that her most important concerns are about what her students are learning.

In Mrs. Clark's classroom, students teach themselves to think mathematically and to apply their understanding to new situations. Mrs. Clark seems to have tapped an internal motivation within the students to be responsible for validating their ideas. By not serving as the authority for deciding the correctness of answers, she projects this authority onto the students. Through negotiating with the students in her class, Mrs. Clark provides them the opportunity to prove the validity of their answers and know when their answers make sense. Mrs. Clark says that she wants the students to become responsible for their own thinking, not to depend on her or another teacher for their encouragement. As the year progressed, it became obvious that the students did not consider Mrs. Clark to be the sole authority for knowing the correct answers or the correct methods of finding answers. Many of the examples of students' explanations demonstrate that the students in this fourth grade have taken responsibility for their learning and have begun the journey toward conceiving mathematics as "a way of thinking."

Relationship of Findings to Previous Studies

Constructivist learning theory implies that individuals actively construct knowledge through a process of conceptual change by assimilating and accommodating new knowledge with prior knowledge (Piaget, 1970, 1980). Some researchers using assumptions of constructivist learning theory have investigated how children construct specific mathematics concepts (Carpenter & Moser, 1984; Cobb, 1988; Hiebert & Wearne, 1985, 1986; Steffe & Cobb, 1988). Other researchers (Carpenter & Fennema, 1991; Carpenter et al., 1989; Fennema & Carpenter, 1992; Fennema et al., 1989) have studied how giving teachers knowledge of how children learn mathematics influences their teaching. Carpenter, Fennema, and colleagues found that knowledge of how children learn mathematics definitely affects how teachers design instruction. Cobb, Yackel, and Wood

(1989) designed instructional materials and guidelines about using the materials with cooperative grouping of students. Cobb, Yackel, and Wood (1990, 1991, 1992) observed how using these materials and new ideas forced teachers to examine their thinking about their children's learning. Cobb et al. (1990, 1991) found that teachers do change their instruction when they are confronted with the thinking of their students. Kamii (1985, 1989) investigated and documented the interactions in the classroom between teachers and students during mathematics lessons. Kamii found that if teachers do not impose procedures on students, they will naturally develop their own algorithms for mathematical concepts. Lampert (1989, 1990, 1991, 1992) has shown us how a practitioner researcher goes into an elementary classroom and teaches children to become mathematical thinkers. However, her research does not show us what happens when a "real-life" elementary teacher's teaching practice coincides with constructivist theory. What is the teachers' thinking? What dilemmas does the teacher encounter? What decisions are made? How does a teacher interact with students to build lessons on students' thinking? Researchers and mathematics educators have asked teachers to change their teachers toward an approach that is based on constructivist learning *theory* without giving teachers an appropriate picture of what this approach to teaching *practice* looks like when real teachers try it.

My study expands the knowledge base of constructivist teaching and learning in mathematics. It provides us with knowledge of what approaches to teaching that use constructivist learning principles looks like in practice. It portrays how a real-life teacher interprets and implements *The Standards* without the help of "experts." I observed one teacher every school day for one semester, not just once a week or one week out of a month. I was able to examine all aspects of the teacher's instruction and also assess the learning environment. The length of time and the amount of rich data that I collected and analyzed helped me to obtain understanding of the depth and breadth of this teacher's thinking about instruction and classroom practice. I was able to ask her not only about her conceptions of mathematics and mathematics teaching and learning but also to observe how or whether these conceptions played out in her decision making and actual teaching practice. I was able to identify and analyze the strategies of one teacher's approach to constructivist teaching. My research indicates that this type of teaching has a character of its own. It is more than strategies. It is a total learning environment where students collaborate with the teacher to create their own learning. It is a way of thinking about mathematics teaching and learning that is solidly grounded in that teacher's philosophy and background.

Unlike other researchers that I have discussed, I did not provide this teacher with materials or ideas (Cobb et al., 1990, 1991, 1992) nor with knowledge of how children learn (Carpenter & Fennema, 1991, 1992; Hiebert & Wearne, 1985, 1986, 1991). I did not ask her to reflect on her teaching in order to get her to change her thinking or behavior. I simply observed this teacher as she taught. Perhaps one of the most important observations that I made was that this teacher is actually constructing her own knowledge of constructivist learning theory while assimilating and accommodating *The Standards* within her teaching.

Implications for Future Research

In order to gain a better understanding of how teachers use constructivist approaches, much more remains to be learned about what this teaching "looks like." From the present study, I make four recommendations for future research. First, it is crucial to have more models of teachers using constructivist approaches to teaching mathematics. Second, the current study suggests that there is a need for more narratives about teachers who are changing their mathematics teaching. Third, this study illustrates the need for long-term qualitative studies of teachers. Fourth, it is very important to learn more about how universities and teacher education programs can help preservice and inservice teachers learn how to teach using constructivist learning principles.

Recommendation 1

We need replications of the present study in order to understand other elementary teachers' approaches to mathematics teaching and learning. The present study identifies 10 strategies consistent with constructivist learning theory that one teacher uses. Perhaps another teacher would use different strategies or use the ones that I have identified in different ways. More models of mathematics teaching that are consistent with constructivist learning theory are needed. Would all models look similar? Would a teacher with different conceptions have similar approaches? Would a different group of students make a difference? Would the grade level make a difference?

Recommendation 2

Teachers need to share their own experiences about implementing "constructivist teaching." Teachers need to talk about their teaching. This sharing can help others predict and better understand the experiences they might encounter. We need teacher narratives of teachers who are struggling with dilemmas in their own classrooms when using constructivist learning approaches. It is important to document how other teachers have constructed their definitions of constructivist teaching. There is clearly not one way. These narratives could help teachers examine their beliefs about the ways children learn mathematics and how these beliefs can reflect on the learning opportunities they provide their students.

What makes a teacher use *The Standards?* Must teachers accept the learning philosophy that is the basis for the reform movement of mathematics before they can change their teaching? I believe that more investigation of questions like these will help teachers, teacher educators, and researchers work together to help create classrooms where students become mathematical thinkers.

Recommendation 3

This study illustrates the need for more long term qualitative studies of teachers. It was difficult to study and assess teacher thinking. I found that I best understood thinking about instruction by observing this teacher over a long period of time rather than by asking her specific questions about why she made decisions. Teachers have difficulty in accessing their own thinking or explaining why they make certain decisions. Researchers need to be there day to day and observe each teacher. To go once a week or once a month as many mathematics researchers have done does not provide a complete picture of the teaching.

Recommendation 4

Teacher education programs and universities need to investigate ways to help preservice and inservice teachers develop teaching strategies that are consistent with constructivist learning principles. Universities faculties might ask questions such as: How can teacher educators develop programs that help preservice and inservice teachers better understand constructivist learning principles? More research would help us understand how constructivist approaches to teaching can be developed. Current teacher education researchers believe that methods courses should be taught by educators who model approaches that are constructivist learning oriented (Ball, 1988, 1991; Brown, Cooney, & Jones, 1990). Would these types of experiences help teachers and preservice change their conceptions about mathematics teaching and learning? Would these types of experiences help teachers understand what the practice of using constructivist approaches to teaching can look like?

A second area of future research might be--how universities can help schools accommodate the NCTM *Standards* into their mathematics curriculum. If teacher educators and researchers become more involved in working with teachers by conducting workshops about how to teach so their students construct their own knowledge, would teachers better understand how to implement *The Standards* ? When schools try to implement curriculum that is based on new approaches to learning theory, teachers need to be supported by teacher educators with whom they can consult. Similarly, teacher educators would profit from visit to schools, observations, and collaboration with teachers who are implementing *The Standards* (e.g. constructivist learning theory). This collaboration will allow teachers and teacher educators to gather more information about how constructivist principles work in classrooms.

Last, we need more research about how teachers who are working hard to implement *The Standards* in their classrooms can help other teachers do the same. School districts need to find teachers who are trying to implement *The Standards* and support them in helping other teachers. What one teacher builds in one year to create a dynamic learning community may be torn down by the next teacher. For instance, one of Mrs. Clark's students, Kevin, had come into her fourth grade last year not liking mathematics. By the end of the year, however, his attitude had changed. He said that he "liked math now." This year, after about a month in the fifth grade, Mrs. Clark said he stopped in to tell her that "math wasn't his favorite subject again." What one teacher had helped build with him had been diluted by his new experience in the fifth grade. Perhaps the worst part of this is that he has projected his negative experiences onto the subject matter itself. Unfortunately, we know from the research of others that he may not voluntarily take mathematics again (Battista, 1986; Martinez, 1987; Wigfield & Meece, 1988).

Implications for Teaching

I focus on five issues that seem to be central to reform in teaching mathematics. I suggest recommendations about the purpose of mathematics teaching, about curricular reform, about learning environments, about teaching the *individual* student, and about time dilemmas.

Recommendation 1

The primary purpose for mathematics teaching should be the development of autonomy in thinking. What do students make of what they are doing? Data from this research suggest that the students are learning what the teacher is teaching--mathematics is a way of thinking. These results demonstrate that when students are given opportunities to learn to think mathematically, they become divergent thinkers, deeper thinkers. Mrs. Clark's students became quite good at finding mathematical relationships and patterns. The explanations of their thinking provide evidence that students constructed powerful understandings of the connections of concepts and procedures. Not only did the students learn mathematics, but they also developed positive attitudes about mathematics and about themselves in being able to think about mathematics. They were enthusiastic when attempting new problems or activities.

Education should no longer be about students trying to figure out the right answers--the answers that the teacher wants. Students should not be made to think that they will get into trouble if they do not give the *one* correct answer. Technology and change is occurring so rapidly that it is difficult to keep up with the new information. Therefore, it is important not just to know the correct product or answer but to know the thinking processes involved in getting that correct product (answer).

Findings from my research show that students who learn to think mathematically and value their thinking became autonomous learners. Kamii (1991) also found that students who learn to think logically and make decisions about approaches to their own solutions to problems develop more autonomy about their own learning than students who learn mathematics by memorizing and applying rules. Kamii found, as I did, that if students learn to value their thinking and trust their decisions, they build confidence and empower themselves.

In Mrs. Clark's classroom, the students no longer considered her the sole authority for knowledge in the classroom. Students began to question her. They asked why something worked. Their knowledge became rich with connections and relationships. This richness helped students to reason and use the concepts to apply procedures. Students often suggested alternatives and sometimes even disagreed with the teacher.

However, even though teachers should not be considered the only authority for correct answers, teachers must not give up the authority that comes justifiably from their knowledge of the content and their responsibility to teach that content. Mrs. Clark always walked a fine line between allowing students to present their own thinking strategies and insuring that they were constructing *correct* mathematical knowledge. Therefore, she became the knowledgeable expert who had the authority to create an environment where students constructed correct knowledge (Vygotsky's (1962) idea of the zone of proximal development). Students should not construct mathematics in an environment that is not guided by a knowledgeable teacher. They should not be left to make sense of problem situations and activities without the teacher telling them *anything* (Cobb et al., 1992).

Recommendation 2

Mathematics textbooks should not contain page after page of procedural computations but should contain significant problems central to primary concepts in mathematics. One of the most significant factors in helping students to learn to think and value mathematics is the tasks or activities in which they are involved. Teachers should involve students in activities that encourage them to construct their thinking at deep levels. Teachers should ask students to reason and use their reasoning to build theories that they could prove. Students should be engaged cooperatively and individually in exploratory lessons involving hands-on tasks. These activities should be open-ended, challenging, and problematic for students. Teachers should find mathematical problems in everyday experience that are relevant for students. Teachers should use a few significant problems each day to teach the lesson. These problems should focus on the primary concepts that teachers want students to learn and to relate to their previous learning.

The research that I have reported in this study can be informative to curriculum developers. We can learn from Mrs. Clark's approach that the development of textbooks with page after page of practice problems is a poor way to learn mathematics. Textbook publishers need to develop texts in a very different way. For instance, one unit could have six significant problems that teachers can use to develop their lessons. These problems should be rich so that solving them will foster *quality* discussions. When teachers see texts with fewer problems about primary concepts instead of rows of problems for students to calculate, they might begin to change their instruction. To create a worthwhile classroom environment, a teacher must use concrete problem-solving contexts where students learn to verbalize their thinking.

Recommendation 3

Teachers must show that they value students' thinking. The learning environment in which students learn affects how they learn mathematics and what they think of mathematics (Cobb et al., 1989, 1990; Nickson, 1992). To encourage students to express their ideas, a teacher must show that he or she values students' thinking. Teachers must listen closely to what students say and encourage them to take risks. Classroom communication and interaction are primary issues for teachers to consider in constructing positive non-threatening learning environments (Cobb et al., 1990). My research shows that listening is important. Teachers must listen closely when students tell how they solve problems or what they are thinking. This example of careful listening will provide a model for students to follow. By taking the students' ideas and thinking seriously, teachers show students that their ideas and thinking is valuable. Teachers should encourage students to take risks by not making judgments about correct or incorrect answers.

Recommendation 4

Constructivist teaching should be about teaching to *individual* students. Teaching with a constructivist approach to learning is teaching to individuals. Constructivist teachers know how to teach individuals through whole-group discussions. Mrs. Clark talks with students as individuals even when the organization for the lesson is whole-group instruction. These teachers observe individual students interacting and cooperating with other students. They listen to individual students explain their thinking. When she interacts with individuals, she does not leave out other students. Mrs. Clark also plans with individual children in mind. She anticipates difficulties that children will have in learning certain topics or content.

Every school situation is different. Every class is different. Every child is different. The nature of each classroom evolves its own culture. The teacher who uses strategies that focus on helping students construct mathematical power has to recognize the differences with each class of students and adapt instruction to fit this culture. Mrs. Clark says, "When you do something and you don't get the results that you want, then you change. You start with certain things and get them going in some direction." Mrs. Clark talks about the difference in this year's class of students compared to the class that I observed last year. She says, "They are not as deep thinkers as the students that I had last year. However, they really enjoy making up their own word problems." Therefore, Mrs. Clark adapted her instruction to her students. She has recently changed the fifth section of mathematics class. She calls it "Make Up Your Own Problem." She gives students a certain topic about which to make up a problem. They have learned so much from making up their own problems that she has decided to help them publish a book of their word problems. They will be giving copies of the book to other fourth grades in the county. She is using the development of this book not only for helping them to learn new concepts but also for reviewing for the CTBS test in the spring.

Recommendation 5

A teaching approach based upon constructivist learning theory takes time. One of the greatest implications for teaching that is based on constructivist learning theory is that *it takes time* for students to develop conceptual understanding. *It takes time* to engage students with tasks that will interest and challenge them to solve and construct understanding of mathematics. *It takes time* to establish a learning environment where students feel safe in taking the risks to participate by contributing their thinking. *It takes time* for teachers and students to learn to interact with each other in constructive ways to achieve a cooperative community of mathematical thinkers. Mrs. Clark says that in the beginning of the year, her students did not want to talk about how they solved problems. She says that it took time for them to trust her. Over time they began to understand that their thinking about the answer was more important than the answer. A teacher must make difficult choices about what to emphasize and how long to proceed with certain content.

Teachers must be patient in taking the time to make the changes that allow a learning environment to evolve where students learn to think and value their thinking.

Recommendation 6

Teachers should have mathematical power in their own thinking. Teachers must be confident in their own mathematical knowledge and thinking power. Mrs. Clark demonstrates with the tasks that she chooses a coherence of mathematical knowledge and uses this knowledge to help her students learn mathematics. As Mrs. Clark works with students, she shows her own mathematical thinking power. Lappan (1993) noted that teachers need to be role models who demonstrate that mathematics is learned through multiple paths and false starts. Throughout her teaching, Mrs. Clark gives the students the opportunities to become more aware of the thinking process involved in learning mathematics for understanding. Teachers need to understand the process and the mathematics they are teaching. Students need to create mathematics with a knowledgeable teacher.

Teachers must become more knowledgeable about what actually demonstrates mathematical understanding. How do individuals develop theories, create plans for proving them, analyze their plans, and justify their conclusions? Teachers should be able to extend their thinking to new situations.

Teachers need to learn ways of alternative assessment. They need to learn how to choose tasks that encourage discussion and cooperative-group problem solving. Teachers need to learn how to assess the knowledge of the *whole* student's learning of mathematics. They need to learn to construct meaningful authentic assessments that are more challenging and informative than written tests. Teachers need to explore and design new ways to assess students that give *all* students opportunities to show what they understand.

Closing Remarks

Mathematics educators have asked the question, "Is it possible for elementary teachers to teach within the guidelines of constructivist learning suggested in *The*

Standards?" The findings in my research suggest that it is possible for an elementary teacher to develop and implement instruction based on the constructivist learning theory. Mrs. Clark has internalized *The Standards* to such a degree that she automatically reflects upon her own teaching and the learning of her students with constructivist learning principles in mind.

Mrs. Clark's kind of teaching is not typical of elementary mathematics teaching. Her approach is more demanding than traditional direct instruction approaches. Since constructivist learning is based on conceptual learning, it is more challenging and causes teachers to continually reflect upon the learning of students. I do believe, nevertheless, that instruction like hers can be implemented by all teachers *if they have support.* Many teachers have the potential to make changes in their teaching like those suggested in the NCTM *Standards.* The more a teacher is involved in teaching mathematics to facilitate students constructing their own knowledge, the better the teacher becomes. Instruction builds upon instruction much as the "constructivist teacher" builds upon students' knowledge. The more a teacher practices this type of teaching, the better he or she becomes at listening and encouraging students, at making decisions with students' learning in mind, and at facilitating students to actively construct powerful mathematical ideas. Mrs. Clark's teaching encourages us to increase our understanding of *what happens when a teacher implements constructivist teaching and learning of mathematics in an elementary classroom.*

APPENDIX A
MATHEMATICS BELIEFS SCALES

On the following pages is a series of sentences. You are to mark your answer sheets by telling how much you agree that the statements are true.

As you read the sentence, you will know whether you agree or disagree. If you strongly agree, circle A opposite the appropriate number on your answer sheet. If you agree, but not so strongly, or you only "sort of" agree, circle B. If you disagree with the sentence very much, circle E for strongly disagree. If you disagree, but not so strongly, circle D. If you are not sure about a question or you can't answer it, circle C.

Do not spend much time with any statement *but be sure to answer every statement*. Work fast but carefully.

There are no "right" or "wrong" answers. the only correct responses are those that reflect what you believe to be true. Be sure to respond to each item in a way that reflects your beliefs.

THIS INVENTORY IS BEING USED FOR RESEARCH PURPOSES ONLY AND NO ONE WILL KNOW WHAT YOUR RESPONSES ARE.

DO NOT MARK ON THIS BOOKLET.

Developed under a grant from the National Science Foundation.

1	2	3	4	5
A=Strongly Agree	B=Agree	C=Undecided	D=Disagree	E=Strongly Disagree

1. Children should solve word problems before they master computational procedures.

2. Teachers should encourage children to find their own solutions to math problems even if they are inefficient.

3. Children should understand computational procedures before they spend much time practicing them.

4. Time should be spent solving simple word problems before children spend much time practicing computational procedures.

5. Teachers should teach exact procedures for solving word problem.

1	2	3	4	5
A=Strongly Agree	B=Agree	C=Undecided	D=Disagree	E=Strongly Disagree

6. Children should understand the meaning of an operation (addition, subtraction, multiplication, or division) before they memorize number facts.

7. The teacher should demonstrate how to solve simple word problems before children are allowed to solve word problems.

8. The use of key words is an effective way for children to solve word problems.

9. Mathematics should be presented to children in such a way that they can discover relationships for themselves.

10. Even children who have not learned basic facts can have effective methods for solving problems.

11. It is important for a child to be a good listener in order to learn how to do mathematics.

12. Most young children can figure out a away to solve simple word problems.

13. Children should have many informal experiences solving simple word problems before they are expected to memorize number facts.

14. An effective teacher demonstrates the right way to do a word problem.

15. Children should be told to solve problems the way the teacher has taught them.

16. Most young children have to be shown how to solve simple word problems.

17. Children's written answers to paper-and-pencil mathematical problems indicate their level of understanding.

18. The best way to teach problem solving is to show children how to solve one kind of problem at a time.

19. It is better to provide a variety of word problems for children to solve.

20. Children learn math best by figuring out for themselves the ways to find answers to simple word problems.

21. Children usually can figure out for themselves how to solve simple word problems.

22. Recall of number facts should precede the development of an understanding of the related operation (addition, subtraction, multiplication, or division).

23. Children will not understand an operation (addition, subtraction, multiplication, or division) until they have mastered some of the relevant number facts.

24. Most children cannot figure math out for themselves and must be explicitly taught.

1	2	3	4	5
A=Strongly Agree	B=Agree	C=Undecided	D=Disagree	E=Strongly Disagree

25. Children should understand computational procedures before they master them.

26. Children learn math best by attending to the teacher's explanations.

27. It is important for a child to discover how to solve simple word problems for him/ herself.

28. Children should be allowed to invent ways to solve simple word problems before the teacher demonstrates how to solve them.

29. Time should be spent practicing computational procedures before children are expected to understand the procedures.

30. The goals of instruction in mathematics are best achieved when students find their own methods for solving problems.

31. Allowing children to discuss their thinking helps them to make sense of mathematics.

32. Teachers should allow children who are having difficulty solving a word problem to continue to try to find a solution.

33. Children can figure out ways to solve many math problems without formal instruction.

34. Teachers should tell children who are having difficulty solving a word problem how to solve the problem.

35. Frequent drills on the basic facts are essential in order for children to learn them.

36. Most young children can figure out a way to solve many mathematics problems without adult help.

37. Teachers should allow children to figure out their own ways to solve simple word problems.

38. It is better to teach children how to solve one kind of word problem at a time.

39. Children should not solve simple word problems until they have mastered some number facts.

40. Children's explanations of their solutions to problems are good indicators of their mathematics learning.

41. Given appropriate materials, children can create meaningful procedures for computation.

42. Time should be spent practicing computational procedures before children spend much time solving problems.

1	2	3	4	5
A=Strongly Agree	B=Agree	C=Undecided	D=Disagree	E=Strongly Disagree

43. Teachers should facilitate children's inventions of ways to solve simple word problems.

44. It is important for a child to know how to follow directions to be a good problem solver.

45. To be successful in mathematics, a child must be a good listeners.

46. Children need explicit instructions on how to solve word problems.

47. Children should master computational procedures before they are expected to understand how those procedures work.

48. Children learn mathematics best from teachers' demonstrations and explanations.

APPENDIX B
STRUCTURED INTERVIEW QUESTIONS

These questions are those that I pre-planned for interviews. However, these questions often lead to many others that I have not written here.

1. How do you interpret what students are saying?
2. When you are interpreting and rephrasing students' answers, are you trying to get their ideas across or are you trying to help them make arguments?
3. Do you have a purpose in having students be "teachers?"
4. Did you find that at the beginning of school that students were used to this interaction in mathematics?
5. How long does it take to really get them talking and justifying their answers?
6. Have you found that every class becomes accustomed to this interaction?
7. Have you changed the way you approach this type of teaching at all?
8. One day you said, "The answer is not important. What is important is how you are thinking about it." Would you elaborate on that?
9. Do students handle a test in a different way than if they didn't have to learn how to justify their answers?
10. Why are you so emphatic about their having students estimate their answers before they solve them?
11. What are the reasons for your choices of things to measure?
12. Why do you emphasize prediction?
13. Why is it important for students to use correct vocabulary?
14. What mathematics do you see in the calendar activities?
15. How do you distinguish between a students' valid or invalid answer? Both to yourself and to the students?
16. Generally, how do you choose what manipulatives or materials that you are going to use?
17. What is it you are trying to accomplish with the connection between standard and metric measure?

18. What advantages do you see with cooperative groups? What disadvantages?

19. When students are in cooperative groups, how do you evaluate individuals? How do provide for individual accountability?

20. Does everyone in one group get the same grade if they turn in one thing?

21. How do you keep some students from doing more of the work and more of the thinking?

22. How do you choose the problem of the day?

23. Do you like the visualization problems? I notice you use them quite a bit.

24. Why do you teach the divisibility rules?

25. How is it you use the notecards that you sometimes have during lessons?

26. Do you over plan? Why?

27. Why is it important for a student to rephrase another student's answer?

28. Do you use students' interaction in class as part of their grade?

29. Do you use pretests?

30. What is the purpose of introductory lessons?

31. What do you call this kind of teaching?

32. What did you mean when you said a lesson can be too hands-on?

33. What is your role during your teaching?

34. How would you tell a new teacher about this kind of teaching? How to plan for it? How to do it during class? How to evaluate it?

35. Why do you think Ann has improved in her thinking and in her interaction?

36. What dilemmas do you see or have you had to deal with in teaching mathematics in the way that you do?

37. How do you choose the appropriate task or problem for a particular concept?

38. What is mathematics to you?

39. What are your goals for your students in learning mathematics?

40. Do you ever have to explain your mathematics teaching to parents?

41. How do you decide which student's response to build the lesson upon?

42. If students come to a way of thinking about some concept that is mathematically incorrect, do you tell them it is incorrect? Who is the authority?

43. What is important about geometric thinking?

44. What is the role of being a student in your class? What do you expect from your students?

45. You said that sometimes you are clear about the direction of lessons and sometimes you are not. When are you clear and when are you not clear?

46. Can you tell me some of the progression in how you got to where you are now in your approach to teaching mathematics?

47. When you think about mathematics and the type of mathematics that you want your students to understand, how do they best learn this mathematics?

48. With the type of mathematics teaching that you practice, what kinds of knowledge do you think a teacher needs to know?

APPENDIX C
PREOBSERVATIONAL INTERVIEW QUESTIONS

(These questions are ones for which I asked the teacher to write answers.)

Preobservational Questions (before two different lessons)

1. Could you tell me what you are planning to do when I observe your lesson?
2. Can you give me some details about what the students will actually be doing?
3. Why did you decide to do this lesson? How does it relate to the rest of your work in mathematics?
4. Is there anything in particular you are hoping to have happen today?
5. How likely is it that this (#5) will happen? What will it depend on? What might upset your plan?
6. Will this be difficult for any of your students? Why?
7. Is there anything I should especially pay attention to while I am observing?
8. How much preparation time did this lesson take?
9. How will this lesson relate to the outside world? How will it help the students in their real life?
10. Were you prompted to teach this lesson because you found the students lacking some ability or knowledge?
11. When you plan your lessons do you like to think through the entire lesson and plan for the kinds of questions you will ask or do you work off the children's comments or questions?
12. Suppose that during the lesson you ask a child a question and he/she hesitates. Will you rephrase the question, ask a similar question, explain the problem, or ask another student?

APPENDIX D
POSTOBSERVATIONAL INTERVIEW QUESTIONS

(These questions are ones for which I asked the teacher to write answers.)

Postobservational Questions (after two different lessons)

1. How do you think things went in class?
2. How did things compare with what you had expected?
3. Was there anything you were particularly pleased about? What? Why?
4. Which of these (#3) will you draw on in future teaching experiences?
5. Did anything disappoint you? What? Why?
6. What would you modify in future lessons like this one?
7. How did you decide whom to call on? (to work at the chalkboard, to answer questions, etc.)
8. How did you choose the problems in today's lesson?
9. Was there any reason why you chose these problems?
10. How could you improve your teaching performance of this lesson?

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BIOGRAPHICAL SKETCH

I was born and raised in Rainelle, West Virginia. During the first 18 years of my life, I lived in the same house in Rainelle. However, since leaving there to go to college, I have lived in many areas of the United States--northeast, south, and midwest.

I received the Bachelor of Arts degree in elementary education at Morris Harvey College in 1973. During the time I was completing this degree, I also married Paul Steele. I taught as an elementary school teacher for 13 years while I pursued a Master of Arts degree. I received the Master of Arts degree from West Virginia University in 1985. A year later I completed the Bachelor of Science degree in mathematics from the University of Charleston. While I was completing the latter degree, I taught in a secondary private school in Charleston, West Virginia.

In 1986 my husband and I moved to Evansville, Indiana, where I began working as an academic advisor and mathematics instructor at the University of Southern Indiana. I began my doctoral work at the State University of New York at Buffalo in 1990 while teaching at Buffalo State College. I will receive the Ph.D. in instruction and curriculum from the University of Florida in 1995.

I will serve as an assistant professor in the Curriculum and Instruction Department at the University of Nebraska-Lincoln beginning in August 1995. I will be teaching elementary mathematics methods and developing a middle school mathematics education program.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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This dissertation was submitted to the Graduate Faculty of the College of Education and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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